COMMUNICATION BETWEEN SHAREHOLDERS

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ABSTRACT. We study an *expert shareholder* who chooses how much information to communicate about a potential investment's return to a *controlling shareholder* who controls the investment strategy. We embed this model into two settings. In the first setting, equity ownership is determined by investors buying shares on a competitive equity market. We provide conditions under which share-trading delivers perfect communication and full risk-sharing. However, an inefficiency lurks in the background: a competitive market for shares fails to reward the positive externality that the expert investor provides to other investors by purchasing shares. The second setting is a principal-agent relationship where the equity is granted as compensation by a principal (board) to an agent (CEO). Within this setting, the model highlights a novel trade-off between incentivizing effort provision and promoting information transmission.

1. INTRODUCTION

This paper presents a model of information transmission by one shareholder to another. The setting is designed to study how the incentives to transmit information depend on the amount of equity owned by shareholders. In the model, after shares are assigned one of the shareholders learns some information about the return of a risky action (e.g., how much to invest in a risky venture). He is referred to as the *expert investor.* Once the expert investor becomes privately informed he can no longer trade on this information, but he can

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transmit information to a *controlling investor* (she) who chooses the risky action. After the communication takes place, the controlling investor chooses an investment level and payoffs are realized.

We embed this information transmission model into two settings. In both settings shares are obtained (purchased, or assigned) anticipating the communication stage that follows. In the first setting, shares are obtained by buying shares on a competitive equity market. We find that the incentives to trade shares for risk-sharing can be remarkably aligned with the incentives to trade shares for maximal information transmission. If all investors acquire shares by trading in a competitive equity market, then in any competitive equilibrium information can be transmitted perfectly, and moreover there exists a competitive equilibrium that achieves the efficient risk allocation (Section 5.1). However, an inefficiency lurks in the background: a competitive market for shares fails to reward the positive externality that the expert investor provides to other investors by purchasing shares. In some scenarios, this externality leads the expert investor to acquire too few shares and to transmit too little information to the controlling investor (Section 5.2). This inefficiency may be a signficant applied contribution as it points to a conflict between the risk-sharing and the information-transmission motives for trading. This conflict may also arise if the expert investor has a shorter-term horizon than the controlling investor: see Section 5.3. The expert investor can be interpreted as an "activist investor."[1]

The second setting we consider is a principal-agent relationship where the shares are granted as compensation by the controlling shareholder (the principal, for example a corporate board) to the expert shareholder (the agent, for example a CEO). The agent provides information to the principal regarding the principal's actions. In addition, the agent also exerts costly non-contractible effort. Thus the setting features information transmission as

[1]Activists investors feel they have valuable information regarding business strategy and are eager to advise corporate boards. A challenge for activist investors has been to persuade the shareholders who control the board, which are often large institutional investors, to follow their advice. To be persuaded, the controlling shareholders claim that they need to see "skin in the game," i.e., the payoff-alignment (or lack thereof) that equity ownership is thought to provide. Similarly, activist investors say that they want the board to have "skin in the game." Consistent with the importance of "skin in the game," in our model the activist investor's ability to persuade the controlling shareholder depends partly on the alignment provided by their equity holdings.However, our model abstracts from another channel through which some activist investors seek to influence strategy: by obtaining representation on the board or a proxy vote on their issue.

well as moral hazard and risk sharing. Equity can be used as part of the agent's compensation scheme to incentivize two different activities: effort provision and information transmission. We ask whether, in either setting, the equilibrium share allocation will feature full risk-sharing and perfect communication. We find that the optimal contract allocates too much equity to the agent for full communication to occur (Section 5.5). At the optimal contract the agent will feel that the principal's investment strategy is too aggressive and this will lead the agent to misreport his information. In addition, as usual in a principal-agent problem, too much risk (equity) is provided to the agent relative to the first-best risk sharing allocation. We conclude by briefly discussing the scenario in which trading takes place after the expert investor has received the information (Section 5.4).

This paper makes two main contributions to the cheap-talk literature: first, that the shareholdings drive the incentive to communicate; second, in that equity ownership is determined endogenously, hence the sender and receiver make choices before a cheap-talk game with direct implications for their alignment of interest. Technically, our cheap-talk framework builds on a specific strand of the literature including Admati and Pfleiderer (2004), Alonso, Dessein and Matouschek (2008), Kawamura (2015), and especially Alonso (2009). Conceptually, two seminal cheap-talk papers are related to our work. Gilligan and Krehbiel (1987) demonstrate that a controlling agent may want to partially delegate/reduce their ability to exercise control, in order to motivate the sender to acquire information. Dessein (2002) shows that it can be in the interest of a controlling agent to delegate control (in his terminology, authority) to an informed agent, in order reduce the informational loss caused by cheap talk. In both papers, the focus is on the *allocation of control* between sender and receiver. In our paper, in contrast, the focus is on *allocation of ownership.* Thus our paper explores a novel tool to achieve transparent information transmission. Whereas the previous literature may be more relevant in environments where it is difficult to manipulate ownership allocation (political settings, for example), our analysis may be more relevant in environments where ownership can be flexibly adjusted (as in publicly traded corporations, for example), but delegation may not be possible (boards have fiduciary responsibilities).

The corporate boards literature counts several papers based on cheap talk. Adams and Ferreira (2007) analyze the consequences of the board's dual role as advisor as well as monitor of management. Harris and Raviv (2008) analyze the dynamics between inside and outside directors. Adams *et al.* (2010) provides a review of this literature. Two papers deserve special mention. The first is Song and Thakor (2006), which is related for two reasons: first, though the CEO does not technically engage in cheap talk, she does communicate with the board; second, like in our model, the CEO advises but the board decides. This paper studies the CEO's equilibrium communication with the board. The most closely related paper is Almazan et al. (2008), which shares some similarities with our analysis of the principal-agent problem (Section 5.5). In this paper a manager is hired by a board and offered a contract that is designed to incentivize two managerial tasks: costly effort, which increases the probability of good outcomes; and subsequently information transmission (cheap talk) by the manager to investors about the realized outcome. Thus, both Almazan et al. (2008) and our paper study the provision of incentives for the dual managerial tasks of effort provision and information transmission. The main conceptual difference is that in Almazan et al. (2008) the information is transmitted to investors, and it is used to decide which price to pay for shares; in our paper, in contrast, the information is transmitted to the principal and is used to choose the value-creating (or destroying) action a. Other than Almazan et al. (2008), to our knowledge, no model in this literature allows shareholdings to drive the incentive to communicate via cheap talk; nor are the shareholdings determined endogenously.

In a seminal analysis of a principal-agent setting, Ross (1973, 1974) explored how far the preferences of a principal and an agent can be aligned *and also* risk-sharing be achieved, by utilizing linear incentives. Ross (1974) provides conditions on the utility functions of both principal and agent such that linear incentives can achieve perfect preference alignment and also perfect risk-sharing. Our mean-variance specification for the utility of both principal and agent are a special case of Ross, and accordingly, under certain conditions in our model perfect alignment between principal and agent can be achieved. We return to this issue in Section 5.1.

Our work also relates to a growing literature on shareholders with heterogeneous portfolios and how this heterogeneity affect firm-level decisions regarding competitive strategy: see Azar *et al.* (2017) and Anton *et al.* (2016) for recent papers in this vein. In our model, when investors have heterogeneous portfolios, we find that communication can be misaligned with perfect risk-sharing and that this misalignement may not be solved even when investors are free to trade shares in the enterprise. An earlier related literature going back to Benninga and Muller (1979) and DeMarzo (1993), concerns the degree to which shareholders are able to aggregate their different preferences into a strategic decision, when these shareholders hold different beliefs about the future profitability of current strategic decisions.

A somewhat related literature is the one on monitoring by large shareholders, whose seminal papers include Shleifer and Vishny (1986) and Admati *et al.* (1994); see Gillan et al. (1998) for a review. The monitoring activity in this literature shares some similarity with the communication activity in our paper in that both are value-enhancing activities which produce positive externalities for all shareholders.

2. Model

A firm is contemplating how much capital a to invest in a *new venture*. Investing a yields profits aX. The rate of return X is a random variable with mean μ_X and variance σ_X^2. The parameter σ_X^2 is common knowledge to all investors but μ_X is not.

The firm also has a *mature business* whose profits are given by Y, a random variable with commonly know mean and variance μ_Y and σ_Y^2. The firm makes no decision concerning Y. For analytical convenience we assume that Y is uncorrelated with X.

Aside from owning a fraction of the firm, every investor j may also own a non-tradable, exogenously given *background portfolio* Z_j. For analytical convenience we assume that Z_j is uncorrelated with X.

2.1. Players, Information Asymmetry, Ownership, and Control

A *controlling investor* C owns a fraction $\theta_C > 0$ of the firm and has the right to choose a. She does not know μ_X and has a prior belief that μ_X is uniformly distributed on $[0, \sigma_X^2]$.[2] We will think of the controlling investor as the (median voter in the) board of directors because the board controls the firm's major strategic decisions.[3]

[2]The support's lower bound being zero implies that every project returns *on average* more than the investment; the upper bound being equal to σ_X^2 is just a normalization.

[3]Although the controlling investor's share θ_C will often be "large," it need not necessarily be large for it to for it to entail control rights. For example, if the controlling investor is interpreted as the median investor

An *expert investor* E owns a fraction θ_E of the firm. He knows μ_X. In one setting the expert investor may be an activist investor; in another, a CEO. In both cases, the expert investor has private information and can advocate with the board, but cannot directly choose a. If θ_E is negative the expert investor is a short-seller.

There are a number of *noncontrolling investors* i, who own fractions θ_i of the firm. They too do not know μ_X and their prior on μ_X is uniformly distributed on $[0, \sigma_X^2]$.

2.2. Preferences and Payoffs

All investors have mean-variance preferences with risk-aversion parameter r. Thus, if an agent with background portfolio Z owns a fraction θ of the firm, and the firm invests a, the investor's expected utility equals:

$$\mathbb{E}\left(Z+\theta\left(Y+aX\right)\right)-\frac{r}{2}Var\left(Z+\theta\left(Y+aX\right)\right). \tag{2.1}$$

Note that the background portfolio is independent of θ and thus fixed exogenously: we discuss this assumption in Section 2.6.2 and relax it in Section 5.1. After some algebra, and denoting $\omega \equiv \mu_X/\sigma_X^2$, this expression can be rewritten as:

$$\underbrace{-\frac{\sigma_X^2}{2r}\left(r\theta a-\omega\right)^2}_{\text{new-venture payoff}}+\underbrace{\theta\left(\mu_Y-rCov\left(Y,Z\right)\right)-\frac{r}{2}\theta^2\sigma_Y^2}_{\text{mature-business payoff}}, \tag{2.2}$$

up to an additive constant which does not depend on θ or a.[4] The investor's payoff can be decomposed into the portion originating from the new venture aX, and the portion originating from the mature business Y. Only the first portion depends on the controlling investor's strategy.

Henceforth, we normalize the controlling investor's risk parameter $r_C = 1$ and denote the expert's and the noncontrolling investors' risk parameters by r_E and r_i, respectively.

in the board, that investor's share could be small. Alternatively, the controlling investor could be a founder endowed with relatively little "supervoting stock."

[4]Note that this step makes use of the assumption that X is uncorrelated with both Y and Z. The algebra is shown in Lemma B.1 in an online appendix.

2.3. Information Transmission

Before the controlling investor chooses the investment level a, the expert investor may transmit information to the controlling investor through a single round of cheap talk.[5] This cheap talk stage takes place under common knowledge of all the investors' share holdings.

2.4. Acquistion of Shares

Before the cheap talk stage, shares are acquired by all investors (for example, by trading in a competitive market, or as compensation in an employment contract) and the investors' holdings become known to all.

2.5. Timing of Events

(1) All investors acquire shares in the firm.
(2) The expert investor learns μ_X.
(3) The expert investor engages in cheap talk.
(4) The controlling investor directs the venture to invest a.
(5) X, Y, and the Z_j's are realized and payoffs accrue.

2.6. Discussion of Modeling Assumptions

2.6.1. **Roles of Ownership In the Model.** In our model, equity ownership serves three roles. First, it is a source of wealth and risk for all investors. Second, equity represents "skin in the game" that the expert investor needs to have in order to credibly transmit information to the controlling investor; how much "skin" is needed will depend on the controlling investor's risk aversion and equity holdings. Third, equity ownership affects the controlling investor's choice of a.

2.6.2. **Heterogeneity Among Investors.** Investors are potentially heterogeneous in two dimensions. First, in their risk-aversion parameter; for given equity holdings, the risk aversion of the expert and controlling investors determine their ability to communicate. Second, investors may differ in their background portfolios. Their background portfolios do not affect

[5]If the noncontrolling investors can also listen in to the cheap talk, nothing changes in the equilibrium. See the discussion in Section 2.6.3. We do not consider multiple rounds of cheap talk.

communication, but the portfolios' correlations with the firm's profits create heterogeneous diversification motives for owning equity in the company. For example, an activist investors' background portfolio may include just a handful of companies, leading to a different appetite for ownership ("skin in the game") compared to a large institutional investor whose background portfolio may be very diversified.

Note that in the baseline model we do not allow the background portfolios to be traded. We remove this assumption in Section 5.1. If the background portfolio are fully tradeable then the efficiency result in Proposition 3 would prevail – refer to the discussion following the proposition. But the tradeable portfolio assumption leads to a somewhat counterfactual equilibrium implication: that all investors would end up holding a replica of the same market portfolio. In reality all investors do not hold the market portfolio, and in particular the implication does not match the stylized fact that activist investors frequently hold different types of portfolios than institutional investors. This is why we choose to make the portfolios exogenous in the baseline model.

2.6.3. **Modeling of Information and Timing of Trade.** We think of the parameter ω as business-specific intelligence that is not public and that can only be generated after examining confidential corporate information (e.g., details about the cost structure). This is why we assume that the active investor cannot trade after learning ω. In Section 5.4 we remove this assumption and explore the case in which trading takes place after the expert investor learns ω. The assumption that μ_X is uniformly distributed is imposed for tractability only, and it is a limitation. Relaxing it may add economic insight, but we stop short of doing so in this paper.[6]

2.6.4. **Source of Friction In Communication.** Friction in the communication game arises solely due to a "misallocation" of ownership. To see this, consider two investors with risk parameters r, r'. If these investors hold shares θ, θ', respectively, in the exact ratio $\theta'/\theta = r/r'$, then their payoffs (2.2) are identical functions of a up to a linear affine transformation. Therefore, at this holdings ratio there is no conflict of interest regarding the optimal choice of a and hence we expect perfect communication.[7]

[6]See Deimen and Szalay (2017) for a model that relaxes the uniform distribution assumption.

[7]In more general settings a judicious choice of θ can help alignment preferences over the action a, but full agreement may not be achieved. To see this, consider a more general form of payoff function

3. Information Transmission with Fixed Equity Holdings

Fix all equity holdings $\theta_C, \theta_E, \{\theta_i\}$. We analyze the game where the controlling investor chooses a after a single round of cheap-talk communication from the expert investor.

Definition 1 (Risk-adjusted holdings ratio)**.** *The ratio $\rho = \theta_C / r_E \theta_E$ will be called "risk-adjusted holdings ratio."*

The risk-adjusted holdings ratio parameterizes the conflict of interest in the communication game. If $\rho = 1$ then there is no conflict of interest, as previously noted in Section 2.6.4. If $\rho < 1$, the conflict of interest arises from the controlling investor holding "too little" equity relative to the expert investor; if $\rho > 1$, the conflict of interest arises from the controlling investor holding "too much" equity. If ρ is negative the expert investor is a short-seller.

The parameter ρ enters expression (2.2) multiplicatively. Cheap talk games with a multiplicative bias parameter have been studied in a number of prior papers, including: Melumad and Shibano (1991); Admati and Pfleiderer (2004); Alonso, Dessein, and Matouschek (2008); Alonso (2009); Gordon (2010); Kawamura (2015); and Deimen and Szalay (2017). The most useful paper for our purpose is Alonso (2009); applying his results, one can show that in our setting there exists a plethora of interval equilibria with partition cutoffs $\{\omega_0, \omega_1, ..., \omega_N\}$, where $\omega_0 = 0$ and $\omega_N = 1$. Upon learning that ω falls into partition (ω_n, ω_{n+1}), the controlling investor's equilibrium action equals $(\omega_{n+1} + \omega_n) / 2\theta_C$. The partition boundaries ω_n solve the following difference equation solve the following difference equation:

$$\omega_{n+2} - 2k_\rho \omega_{n+1} + \omega_n = 0, \tag{3.1}$$

where $k_\rho = 2\rho - 1$. Alonso (2009, Lemma 2) also shows that all equilibria are outcome-equivalent to interval equilibria,[8] and Gordon (2010) shows that if an interval equilibrium exists with N partitions then equilibria with any smaller number of partitions exist; Alonso (2009) shows that all equilibria with a given number of partitions are essentially unique in that they induce the same set of actions. Difference equation (3.1) is well-known because it generates the family of Chebyshev polynomials as its solution.

$\mathbb{E}[u_j(Y + \theta a X)]$,where $u_i(\cdot)$ is some utility function. The dependence of the utility function on j introduces heterogeneity with regards to the preferred choice of a. In this general formulation there may be no choice of θ that can align preferences.

[8]For a generalization of this property, see Gordon (2010, Lemma 1).

Equation (3.1) is homogenous and linear in ω, so if a sequence $\{\omega_n\}$ solves (3.1) then, for any real number α, the sequence $\{\alpha\omega_n\}$ also solves (3.1). This means that the plethora of solutions to the difference equation can be indexed by a single scaling factor. An intriguing possibility is that, by choosing a sufficiently small scaling factor α, one might be able to generate cheap-talk equilibria with an arbitrarily large number of partitions. But this is only possible when $\rho > 1$ (see Lemma 12 in Gordon 2010), and even though the informativeness of the equilibrium grows with the number of partitions, in the limit these equilibria do not approach perfect communication (see Alonso 2009). In these equilibria, low realizations of ω will be communicated very accurately to the controlling investor (receiver), but high realizations of ω will not. In contrast, when $\rho < 1$ the most informative equilibrium has a bounded number of partitions. Proposition A.1 derives all these results within our framework.

The intuition behind equilibria with arbitrarily fine partitions is as follows. When $\rho > 1$ the expert investor's holdings make him more aggressive than the controlling investor, hence the expert wants to misrepresent the return on investment as being higher than it is; this means that the expert is highly credible when he says that returns ω are low. In the limit case where returns are zero the two investors actually agree on the optimal action, which gives rise to the possibility of very fine partitions in the neighborhood of $\omega = 0$. When $\rho < 1$ the expert investor's holdings make him less aggressive than the controlling investor, therefore reports of low ω are less credible. No information at all can be communicated when $\rho < 3/4$. The limit case where $\theta_E = 0$, so that $\rho = \infty$, is noteworthy. In this case the sender has no stake in the outcome and so, as in any cheap-talk game, perfect communication can be sustained in equilibrium.

We now turn to computing new-venture payoffs next. The proof of the next proposition exploits the properties of Chebyshev polynomials, although Part 1 of it could also be derived by applying Alonso's (2009) results.

Proposition 1 (New-venture payoffs in most informative equilibrium)**.** *Fix a risk-adjusted holdings ratio ρ. Equilibrium new-venture payoffs for expert and controlling investors increase in the number of equilibrium partitions. Up to additive constants, equilibrium new-venture payoffs have the following properties.*

(1) *If $\rho \geq 1$, the best equilibrium new-venture payoffs are as follows. For the controlling investor:*

$$\overline{V}_C(\rho) = \frac{\sigma_X^2}{6} \frac{1-\rho}{(4\rho - 1)}.$$

For the expert investor:

$$\overline{V}_E(\rho) = -\frac{\sigma_X^2}{6r_E} \frac{\rho - 1}{\rho(4\rho - 1)} (4\rho - 3).$$

(2) *If $3/4 \leq \rho \leq 1$, the family of equilibrium new-venture payoffs admits the following upper bound. For the controlling investor:*

$$\overline{V}_C(\rho) = \frac{\sigma_X^2}{6} \frac{(\rho - 1)(3\rho - 1)}{(4\rho - 1)}.$$

For the expert investor:

$$\overline{V}_E(\rho) = -\frac{\sigma_X^2}{3r_E} \frac{1-\rho}{(4\rho - 1)}.$$

The exact equilibrium payoffs for the controlling and expert investors are given in the appendix, equations A.6 and A.13, respectively. These bounds are tight in the sense that for any ρ, there exists a value $\overline{\rho} \in (\rho, 1)$ such that $\overline{V}_E(\overline{\rho})$ is attained.

(3) ***(communication breakdown)*** *If $0 < \rho < 3/4$, no information can be transmitted. In this case the controlling investor's equilibrium new-venture payoff equals $-\sigma_X^2/24$ and the expert investor's equilibrium new-venture payoff equals*

$$-\frac{\sigma_X^2}{2r_E} \mathbb{E}\left(\frac{1}{2\rho} - \omega\right)^2.$$

Proof. See Propositions A.2 and A.3. □

Figure 1 depicts the controlling investor's new-venture payoff for different values of ρ. When $\rho < 1$ the actual payoff function is not smooth (orange line, analytical expression provided in equation A.6) because as ρ increases the number of partitions contained in the most informative equilibrium changes discretely; the function $\overline{V}_j(\rho)$ (blue line, analytical expression provided in Proposition 1) provides a smooth approximation to the exact payoff function. From now on we will work with the approximate function.

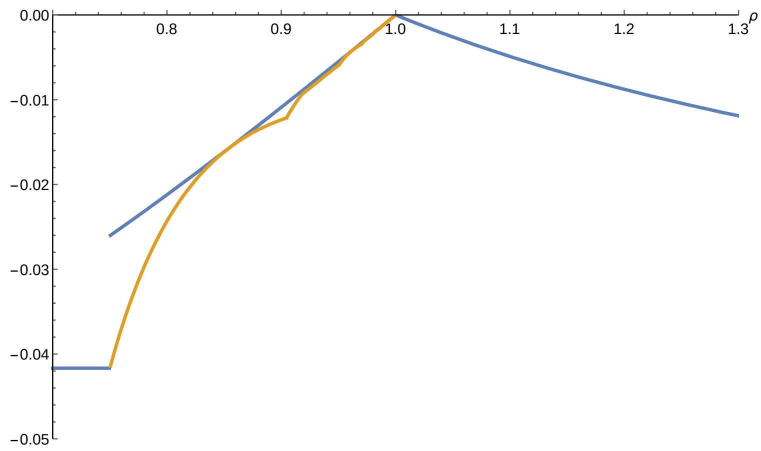


Figure 1. Controlling investor's new-venture payoff for different values of ρ. Note that for $\rho \in \left(\frac{3}{4}, 1\right)$ proposition 1 gives the approximate payoff.

4. Demand for Shares

In this section we compute each investor type's inverse demand for shares. Inverse demand is defined as the marginal utility of increasing the investor's equity holdings. To compute demand it is necessary to take a stand on which equilibrium prevails among the many possible communication equilibria. Consistent with standard practice in the literature on cheap talk, we focus on the most informative equilibrium. This equilibrium is focal because it is the equilibrium that is ex-ante preferred both by the sender and receiver (refer to Proposition 1).[9]

Inverse demand is made up of two components: the portion of demand that reflects the new venture, and the portion that reflects the firm's mature business. We study the first component next, and add the second component later.

4.1. New-Venture Related Demand for Shares

Definition 2. *The inverse demand function for shares of investors $j = C, E$ is the derivative of $\overline{V}_j(\rho)$ with respect to θ_j.*

[9]Enlarging the focus to Pareto-suboptimal equilibria in the communication phase greatly enlarges the set of possible outcomes in the entire game. For example, many equilibrium outcomes in the trading phase can be sustained through the threat of coordinating on a babbling equilibrium in the cheap-talk stage. These equilibria, however, are not renegotiation-proof. In particular, in the subgame following the trading stage, but before nature draws the state variable, both players benefit from renegotiating to playing the most-informative communication equilibrium.

Note that inverse demand is defined based on the function $\overline{V}_j(\rho)$, which, for $\rho < 1$, is only an approximation of the actual utility experienced by investor j.

Proposition 2 (New–venture related demand for shares)**.** *The new–venture related inverse demands for shares are as follows:*

(1) *for the controlling investor:* $\sigma_X^2 \rho (2\rho - 1) / r_E \theta_E (4\rho - 1)^2$ *for* $\theta_C \in \left(\frac{3}{4} r_E \theta_E, r_E \theta_E\right)$, *which is increasing in* θ_C; *zero for* θ_C *smaller than* $\frac{3}{4} r_E \theta_E$; *and negative for* θ_C *larger than* $r_E \theta_E$;

(2) *for the expert investor:* $\sigma_X^2 (8\rho^2 - 8\rho + 1) / 2\theta_C (4\rho - 1)^2 > 0$ *for* $\theta_E \in (0, \theta_C / r_E)$, *which is decreasing in* θ_E; *and negative elsewhere;*

(3) *for* i, *a non-controlling investor:* $\sigma_X^2 (\theta_C - r_i \theta_i) \mathbb{E}\left[a^*(\Omega)\right]^2$, *where* $a^*(\Omega)$ *denotes the equilibrium action taken by the controlling agent when the expert agent employs reporting strategy* $\Omega(\omega)$; *this is a decreasing function of* θ_i.

Proof.

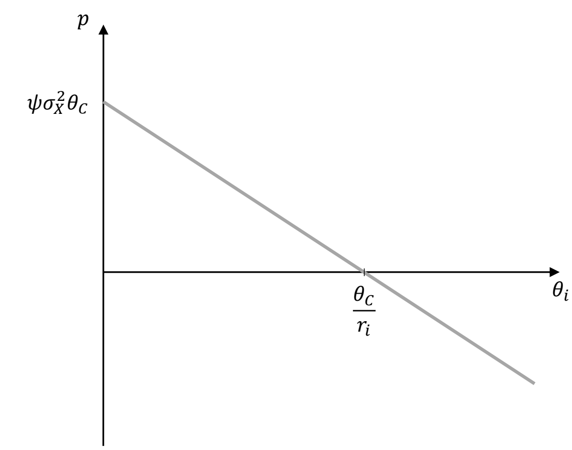
(1) The expression is the derivative of $\overline{V}_C(\rho)$ from Proposition 1 with respect to θ_C. Lemma B.2 and Corollary B.3 in an online appendix show the algebra.

(2) The expression is the derivative of $\overline{V}_E(\rho)$ from Proposition 1 with respect to θ_E. Lemma B.2 and Corollary B.5 in an online appendix show the algebra.

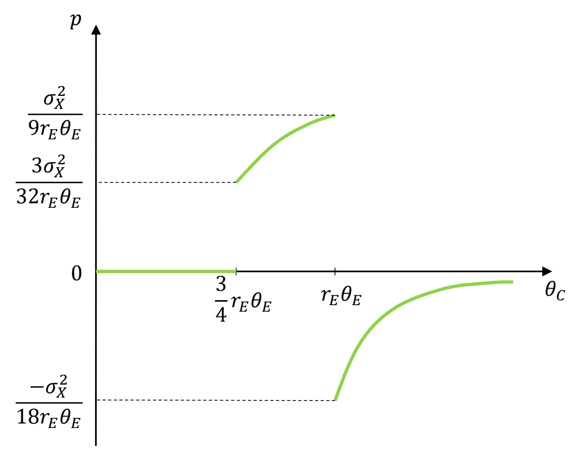
(3) Suppose the controlling investor takes optimal action $a^*(\Omega)$. Then the payoff to noncontrolling investor i reads $-\frac{\sigma_X^2}{2r_i} \mathbb{E}\left[r_i \theta_i a^*(\Omega) - \omega\right]^2$. Differentiating with respect to θ_i and collecting terms yields the result. The steps are provided in Lemma A.4.

□

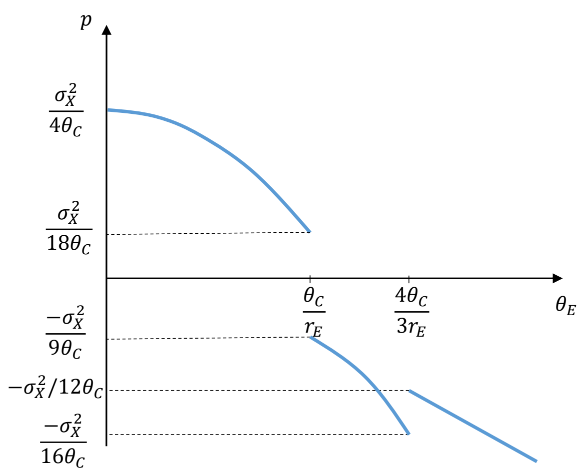
The controlling investor's new-venture related inverse demand is summarized in Figure 2 Panel B. The different regions are noteworthy. First, communication-related inverse demand is 0 when holdings are too low to support information transmission; this is because in this region the controlling investor can costlessly compensate for lower shareholdings by choosing a higher action. When $\theta_C \in \left(\frac{3}{4} r_E \theta_E, r_E \theta_E\right)$ information transmission becomes possible, so demand is positive because adding one more share improves alignment with the expert investor, leading to more communication and a better decision. The discontinuity in demand at $r_E \theta_E$ reflects the informational loss from marginal miscoordination when both investors are perfectly aligned. This discontinuity may give rise to a multiplicity of equilibrium prices supporting a single equilibrium allocation. When $\theta_C > r_E\theta_E$, communication-related inverse demand is negative because every additional share worsens the misalignment with the expert investor, thus worsening information transmission.



Panel A: Non-controlling Investor



Panel B: Controlling Investor



Panel C: Expert Investor

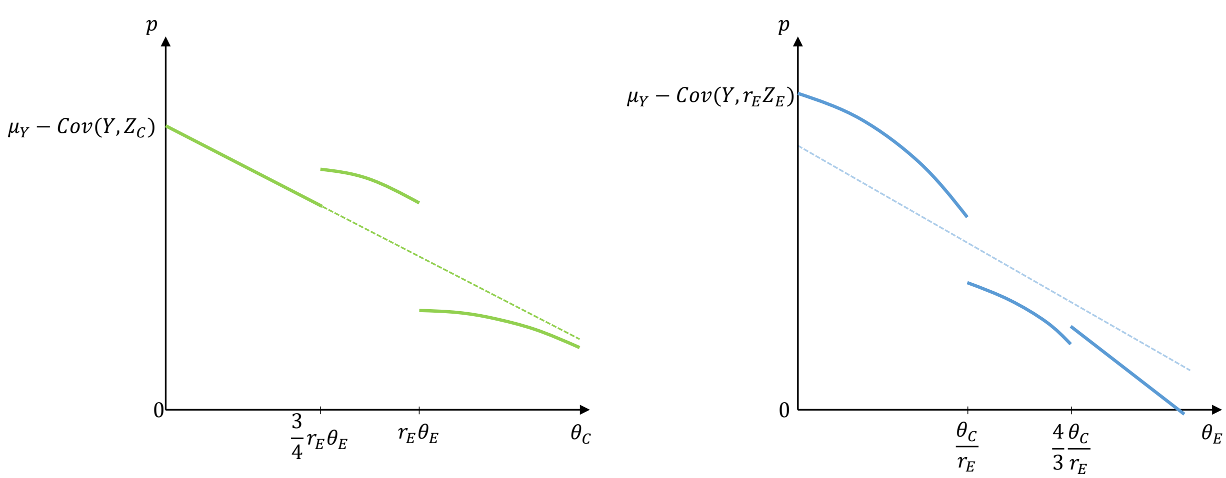
FIGURE 2. Investors' new-venture related demand for shares.

The expert investor's demand is summarized in Figure 2 Panel C. At low holdings demand is positive because exposure to risk through ownership is valuable and, unlike the controlling investor, the cannot simply increase a. In addition, adding one more share improves alignment and communication with the expert investor. When ownership exceeds θ_C/r_E each additional share carries excessive risk and reduces information transmission, hence demand is negative.

A noncontrolling investor's inverse demand is summarized Figure 2 Panel A. This investor takes both the action and the information structure as given, and therefore he has a standard demand for a risky asset: linear and downward-sloping.

4.2. Overall Demand for Shares

The portion of demand that reflects the firm's mature business is obtained by differentiating with respect to θ_j the mature-business component of (2.2). That derivative equals $\mu_Y - Cov\,(Y, r_j Z_j) - r_j \theta_j \sigma_Y^2$. This is a standard downward-sloping linear demand for a risky asset and it is depicted in the two panels of Figure 3 by the dashed line. Adding together the new-venture and mature-business components of demand yields total inverse demand.



Panel A: Controlling Investor

Panel B: Expert Investor

Figure 3. Investors' overall demand (solid line) and mature business-related demand (dashed line).

5. Applications and Extensions

This section contains some applications and extensions of the communication model. These applications illustrate a conceptual question: how different share allocation mechanisms shape the sender and receiver's choices before a cheap-talk game with direct implications for their alignment of interest. The following subsections are not intended as self-contained applied contributions.

5.1. Shares Are Acquired On a Competitive Equity Market

This section shows that, if all investors (including notably the controlling agent) acquire shares by trading in a competitive equity market, then in any competitive equilibrium information can be transmitted perfectly, and moreover there exists a competitive equilibrium that achieves the efficient risk allocation.

A competitive equilibrium is a price p of the stock and equity holdings $\theta^*_C, \theta^*_E, \{\theta^*_i\}$ such that, for all $j \in C, E, \{i\}$:

$$\theta^*_j \in \arg\max_{\theta_j \geq 0} E\left[u_j\left(\theta_j; a^*\left(\theta_j, \theta^*_{-j}\right)\right)\right] - p\theta_j - \lambda p\theta_j \text{ subject to } \sum_j \theta^*_j = 1.$$

We denote by $u_j(\cdot)$ the payoff function (2.2), including any background portfolio Z_j; $a^*\left(\theta_j, \theta^*_{-j}\right)$ denotes the equilibrium strategy in the cheap talk game (refer to Proposition A.1 Part 1); and λ is an exogenously specified unit cost of borrowing in order to invest in the venture (interest rate). Note that the parameter λ is assumed equal for all investors.[10] From now on we normalize $\lambda = 0$, meaning that equilibrium prices are measured net of the interest rate. We assume for convenience that $\theta_j \geq 0$,i.e., shorting is not possible.

Proposition 3 (Competitive equilibrium is efficient)**.** *Suppose $Cov\left(Y, r_j Z_j\right)$ is independent of j and all investors acquire shares by trading in a competitive equity market. Then:*

(1) *at any competitive equilibrium $\rho = 1$ and information is perfectly transmitted;*
(2) *there exists a competitive equilibrium where risk is shared optimally among all investors.*

[10]If the parameter λ were investor-specific the trading of shares would not likely result in good risk-sharing properties.

Proof. The necessary condition for a competitive equilibrium reads:

$$\mu_Y - Cov\,(Y, r_j Z_j) - r_j \theta_j \sigma_Y^2 + \frac{\partial \overline{V}_j\,(\rho^*)}{\partial \theta_j} = p \text{ for all } j, \tag{5.1}$$

where p represents the share price.

Part 1. Suppose by contradiction that there exists a competitive equilibrium with $\rho^* \neq 1$. Suppose $\rho^* < 1$; then $\partial \overline{V}_E\,(\rho^*)\,/\partial \theta_E < 0 \leq \partial \overline{V}_C\,(\rho^*)\,/\partial \theta_C$ (with equality holding when $\rho^* < 3/4$; refer to Panel B in Figure 2). In this case (5.1) implies:

$$-\,Cov\,(Y, r_C Z_C) - \theta_C \sigma_Y^2 < -Cov\,(Y, r_E Z_E) - r_E \theta_E \sigma_Y^2. \tag{5.2}$$

Because $Cov\,(Y, r_j Z_j)$ is independent of j, this condition requires that $\theta_C > r_E \theta_E$, which contradicts the premise that $\rho^* < 1$. In case $\rho^* > 1$ we have $\partial \overline{V}_E\,(\rho^*)\,/\partial \theta_E > 0 > \partial \overline{V}_C\,(\rho^*)\,/\partial \theta_C$. In this case the opposite inequality holds in (5.2), which implies $\theta_C < r_E \theta_E$ and thus a contradiction of the premise that $\rho^* > 1$.

Part 2. Consider an ancillary economy where we impose the restriction $a \equiv 0$. In this economy the only tradeable assets is Y, and the first welfare theorem guarantees that the competitive equilibrium allocates the θ_j's efficiently. The competitive equilibrium quantities θ_j^* and price p^* in this economy solve:

$$\mu_Y - Cov\,(Y, r_j Z_j) - r_j \theta_j^* \sigma_Y^2 = p^* \text{ for all } j. \tag{5.3}$$

Because $Cov\,(Y, r_j Z_j)$ is independent of j, these equilibrium conditions yield that $r_j \theta_j^*$ is independent of j. As a consequence $\rho^* = 1$ in this ancillary economy. Therefore $0 \in \frac{\partial \overline{V}_j(\rho^*)}{\partial \theta_j}$, and so the ancillary-economy θ_j^*'s and p^* also solve (5.1). Therefore, the θ_j^*'s and p^* also constitute a competitive equilibrium for the economy where a is not restricted to equal zero. At this equilibrium we know that the risk from the risk bundle $Y + aX$ is shared efficiently because the conditions for efficient risk sharing are that $r_j \theta_j^*$ is independent of j, regardless of whether the risky asset being shares is Y (ancillary economy) or $Y + aX$ (our main model). □

The assumption that $Cov\,(Y, r_j Z_j)$ is independent of j implies that the background portfolios Z_j are "conveniently chosen" for the purpose of risk sharing among investors. To see this, fix the action a. Now imagine that the Z_j's were being assigned to solve the optimal risk-sharing allocation of assets aX, Y, and $\{Z_j\}$. Then every investor j would be assigned a fraction of the aggregate asset $\mathcal{Z} = \sum Z_j$ in proportion to $1/r_j$, hence $Cov\,(Y, r_j Z_j) = Cov\,(Y, \mathcal{Z})$ independent of j. Put differently, if all investors traded their background portfolios Z_j on the share market, then $Cov\,(Y, r_j Z_j)$ would be independent of j. In this case competitive equilibrium in the equity market produces an allocation where risk is shared optimally, information is transmitted perfectly, and the allocation of control rights is irrelevant.[11]

The proof of Proposition 3 indicates that the competitive equilibrium price p^* does not assign any value to the portion of risk coming from the new venture. This feature of the equilibrium is due to the fact that in this model new-venture risk is not scarce: it can be costlessly increased by the controlling agent by increasing a. Therefore the marginal social value of new-venture risk must be zero.

When $Cov\,(Y, r_j Z_j)$ varies with j the trading of shares does not lead to perfect alignment in the incentives to communicate. To demonstrate this, we have worked out a a numerical example involving only an expert investor and a controlling investor with $Cov\,(Y, r_E Z_E) > Cov\,(Y, Z_C)$.[12] The market-clearing price leads to demands for shares such that $\rho = 1.091$. The expert investor's excess of "skin in the game" arises because the expert's background portfolio is *more correlated* with, i.e., less diversified with respect to the venture $Y + aX$ than the controlling investor's background portfolio. This correlation introduces a "diversification" motive for reducing exposure to the venture. This scenario may be of applied relevance for the activist investors case: frequently, controlling investors tend to be well-diversified, whereas informed activists tend to be undiversified.

Another scenario where the presence of background portfolios interferes with the incentives to communicate is the case where the action a creates externalities on firms in the investors'

[11]This finding is connected to Theorem 8.3 in Ross (1974), which provides conditions on the utility functions of both principal and agent such that suitably chosen linear incentives can achieve perfect preference alignment and also perfect risk-sharing. These conditions are satisfied if the two agents have constant risk tolerances. Our mean-variance specification for the utility function (2.1) can be microfounded based on an exponential utility function with constant risk tolerance equal to r. Therefore, using Ross (1974) we know that perfect risk-sharing must imply preference alignment.

[12]The following parameter values are used $r_E = 1$, $\sigma_X^2 = 1$, $\sigma_Y^2 = 1$, $\mu_Y = 1$, $Cov\,(Y, Z_C) = 0$, $Cov\,(Y, r_E Z_E) = \frac{1}{4}$. The equilibrium has the following features: $p^* = 0.386$, $\theta_E = 0.478$, $\theta_C = 0.522$ (rounded to 3 decimal places). In this example $\rho = 1.091$ and thus, while a significant amount of information is conveyed, communication is not perfect. The Mathematica code for the example can by downloaded at `www.kellogg.northwestern.edu/faculty/antic/research.html`.

background portfolios. Suppose for example that the expert and controlling investors both own strategically-connected firms in their portfolios; then their portfolio returns $Z_E(a)$ and $Z_C(a)$ are affected by a. In this scenario, the expert's credibility in communicating information about a will be a function of how closely aligned the random functions $Z_E(a)$ and $Z_C(a)$ are: unless the two portfolios have a large overlap, a conflict of interest with respect to a exists which cannot be fully resolved by trading shares in the venture. This scenario may be of applied relevance in light of recent literature indicating that common ownership patterns can affect the managerial incentives in, and business decisions of competing firms (see Anton et al. 2016, Azar et. al. 2017).

5.2. Firm Has an Entrenched Controlling Shareholder

Activist investors often seek to influence a business' strategy by transmitting information to a controlling shareholder (board, controlling owner).[13] In this setting investor E can be interpreted as an activist investor, and investor C as the controlling shareholder. In this section we assume that the controlling shareholder is entrenched, that is, she has an exogenously given share θ_C of the enterprise. Aside from the controlling shareholder, all other investors acquire shares in the competitive market. We assume $Y = 0$ for expositional simplicity.

In this setting, optimal risk sharing between investors i and C can be especially difficult to achieve: investor C can create risk pricelessly by increasing a, whereas investor i acquires risk at the market price of shares. Therefore, efficient risk-sharing equilibria require zero share price. To allow maximal scope for efficiency, in this section we relax the notion of competitive equilibrium: now, if $p = 0$ we only require that $\sum_j \theta_j^* \leq 1$. This relaxation increases the scope for zero-price equilibria. Despite this relaxation we find that, in contrast with Proposition 3, efficiency will generally not prevail.

Proposition 4 (Inefficient equilibrium when the controlling investor is entrenched)**.** *Suppose that $Y = 0$ and that the controlling investor's holdings are fixed at θ_C. All other investors acquire shares by trading in a competitive equity market. Then:*

[13] In addition, some activist investors also seek to influence strategy by obtaining representation on the board or by seeking to obtain a proxy vote on their issue. We abstract from this important aspect in this paper.

(1) *given a nonnegative share price no investor chooses to hold too much "skin in the game" relative to the controlling shareholder, that is,* $r_j\theta_j^* \leq \theta_C$ *for all* j;
(2) *there is a threshold* $\overline{\theta}_C$ *such that perfect risk-sharing obtains in equilibrium if and only if* $\theta_C \leq \overline{\theta}_C$. *This threshold is an increasing function of* r_E, r_i, *and a decreasing function of the number of noncontrolling investors.*
(3) *there is a threshold* $\overline{\overline{\theta}}_C > \overline{\theta}_C$ *such that information is perfectly transmitted between activist investor and controlling shareholder in any equilibrium if and only if* $\theta_C \leq \overline{\overline{\theta}}_C$. *This threshold is an increasing function of* r_E, *of* r_i, *and of the number of non-controlling investors.*

Proof.

(1) The demand functions in Proposition 2 are negative for $\theta_E > \theta_C/r_E$ and $\theta_i > \theta_C/r_i$, so if price is nonnegative no investor will want to hold more than these amounts.
(2) Perfect communication between controlling shareholder and informed activist takes place if and only if $r_E\theta_E^* = \theta_C$. Perfect risk sharing between controlling and no-controlling agents requires $r_i\theta_i^* = \theta_C$ (refer to condition 5.3) which, by Proposition 2, requires a zero share price. A zero share price is consistent with our expanded notion of equilibrium only if aggregate demand at that price is below 1, that is:

$$\theta_C + \frac{\theta_C}{r_E} + \sum_i \frac{\theta_C}{r_i} \leq 1.$$

Solving for θ_C yields the threshold $\overline{\theta}_C$ and this threshold has the desired properties.
(3) Perfect communication takes place if and only if $\theta_E^* = \theta_C/r_E$; the informed activist is willing to hold this quantity of shares as long as the price does not exceed $\sigma_X^2/18\theta_C$ (refer to Figure 2). If communication is perfect, $a^*(\Omega) = \omega/\theta_C$ and then the non-controlling investor demand equals:

$$\sigma_X^2 (\theta_C - r_i\theta_i) \frac{1}{(\theta_C)^2} \mathbb{E}[\omega]^2 = \sigma_X^2 (\theta_C - r_i\theta_i) \frac{1}{(\theta_C)^2} \frac{1}{3}.$$

Equating this demand functions to $\sigma_X^2/18\theta_C$ and solving θ_i for yields the non-controlling investors' demand at that price, $\frac{5}{6}\frac{\theta_C}{r_i}$. A price below $\sigma_X^2/18\theta_C$ can be an equilibrium price only if aggregate demand at a price of $\sigma_X^2/18\theta_C$ does not exceed

1, that is, if

$$\theta_C + \frac{\theta_C}{r_E} + \sum_i \frac{5}{6}\frac{\theta_C}{r_i} \leq 1.$$

Solving for θ_C yields the threshold $\overline{\overline{\theta}}_C$ and this threshold has the desired properties.

□

Part 1 shows that, no matter how much "skin in the game" the controlling investor has, no other investor acquires more than that when the price is nonnegative (note that "skin in the game" is defined relative to one's risk aversion). The reason is that the controlling investor's "skin in the game" determines her appetite for risk in choosing a, which in turn determines the riskiness of the firm. Other investors anticipate this and buy just enough to achieve their risk-return goals and, in the expert investor's case, to communicate appropriately. Neither of these motives lead investors to exceed the controlling investor's "skin in the game." This finding is consistent with an anecdotal observation that activist investors tend to acquire smaller stakes than the largest permanent controlling investors. Note that the assumption $Y = 0$ is critical for this result, in that if Y was very valuable investors would have additional motives for increasing their holdings beyond the new venture.

Part 2 is not surprising: there is a direct distortion in risk allocation if any investor (the controlling investor, in this case) is assigned too much risk. It is somewhat interesting that, for some parameter values (between $\overline{\theta}_C$ and $\overline{\overline{\theta}}_C$) no additional distorsions (e.g., in information transmission) are present. In this sense, efficient risk-sharing fails more easily than full information transmission.

Part 3 is the most interesting part because of its contrast with the efficiency result in Proposition 3. If demand for risk is sufficiently large, viz., when the controlling investor owns a large share of the enterprise, or when there are many investors and they are not very risk-averse, then the equilibrium share price is too high for the activist to hold exactly θ_C/r_E. Hence $\rho > 1$ and we cannot have full communication. The reason this problem arises is that the controlling investor is not allowed to sell some of her θ_C. If she could, she would sell some θ_C and simultaneously adopt a more aggressive investment strategy a. This processs would bring her holdings down to the point where $\rho = 1$, thus recovering the efficient allocation (Proposition 3).

An intriguing implication of Proposition 4 part 3 is that, in the presence of entrenched investors who hold shares partly to exercise control, the activist investor should be buying more shares from a welfare perspective. This is because the price on the capital market fails to reward the positive informational externality that the activist investor provides by purchasing shares. This implication is intriguing because it would provide an argument for subsidizing share purchases by activist investors. The next proposition makes the argument formal.

Proposition 5 (Welfare gain from subsidizing the activist investor)**.** *Consider the equilibrium allocation when $\theta_C > \bar{\bar{\theta}}_C$. Taking one share away from a non-controlling investor i and allocating it to the activist investor results in a welfare improvement.*

Proof. In the case $\theta_C > \bar{\bar{\theta}}_C$ the equilibrium price p^* is such that the activist investor demands $\theta_E^* < \theta_C/r_E$. Now perturb the equilibrium allocation by taking one share away from a non-controlling investor i and allocating it to the activist investor. The activist's gain from receiving one more share is p^* (this gain takes into account his private inframarginal benefit from improving communication with the controlling investor). The non-controlling investor's marginal loss from giving up one share *given the equilibrium investment strategy* is p^*; and in addition, every non-controlling investor experiences an inframarginal benefit from a *better investment strategy* which results from improved communication between the activist and the controlling investor (that benefit comes from an increase in $\mathbb{E}\left[a^*\left(\Omega\right)\right]^2$, refer to Proposition 2: that this factor is increasing in θ_E is proved in Lemma A.4). Therefore, this perturbation results in a welfare gain for the pair involved in the trade, as well as for all other non-controlling investors. Finally, the entrenched investor also benefits from better communication. Hence, the new allocation is welfare-improving. □

The roots of the inefficiency featured in Proposition 5 lie in the mechanism that governs share allocation. In a capital market the noncontrolling investors are free-riding, in equilibrium, on the expert investor's communicative activity.

5.3. Expert shareholder is short-termist relative to controlling shareholder

It is often argued that activist investors are short-termist compared to large asset managers. This section explores the consequence of investment-horizon heterogeneity between the expert and controlling investors.[14] In this section the enterprise operates for two periods, $t = 1, 2$. The enterprise pays out $Y_t + aX$ in each period. Y_t are i.i.d realizations of Y in period t. Instead, a is chosen and X is realized only once at $t = 1$ and the return aX pays out twice, once in each period. In each period stock in this enterprise can be rented at rate p, which entitles the renter to that period's returns. Investor E is assumed to only rent shares in period 1, whereas investor C is assumed to rent the same amount of shares in both periods; this assumption makes the expert investor short-termist relative to the controlling investor. Neither investor has any background portfolio Z_j. Let primes denote the variables in this new game with heterogeneous horizons.

Fix θ'_E, θ'_C. Investor C receives the random variable $\theta'_C \left(Y_1 + Y_2 + 2a'X\right) - 2p\theta'_C$. Because Y_t and X are statistically independent, the maximization of the utility from $\theta_C 2aX$ is independent of the terms involving Y. A controlling investor with holdings θ'_C can achieve exactly the same value as she would have in the old cheap-talk game by taking half the old action $\left(a' = \frac{1}{2}a\right)$, provided the expert's communication strategy is the same. What expert holdings θ'_E will generate the same communication strategy as prevailed with holdings θ_E in the old game? Based on the recursion (A.3) that generates the communication strategy, these holdings are $\theta'_E = 2\theta_E$. So a controlling investor's new game against θ'_E is the same as the old game against $\theta_E = \theta'_E/2$. Therefore:

$$\overline{V}'_C\left(\frac{\theta'_C}{r_E\theta'_E}\right) = \overline{V}_C\left(2\frac{\theta'_C}{r_E\theta'_E}\right).$$

Investor E receives the random variable $\theta'_E \left(Y_1 + a'X\right) - p\theta'_E$. Given θ'_E, the expert investor can achieve the same value as in the old game provided by the controlling investor plays the old strategy. Since $a' = \frac{1}{2}a$, the holdings that induce the new controlling agent to play the old action are: $\theta'_C = \theta_C/2$. So an expert investor's new game against θ'_C is the same as the

[14] We thank an anonymous referee for suggesting this application.

old game against $\theta_C = 2\theta'_C$. Therefore:

$$\overline{V}'_E\left(\frac{\theta'_C}{r_E\theta'_E}\right) = \overline{V}_E\left(2\frac{\theta'_C}{r_E\theta'_E}\right).$$

Proposition 6. *Suppose short- and long-term investors acquire shares by trading in a competitive equity market. A competitive equilibrium with full information transmission cannot exist if σ_Y^2 is sufficiently large.*

Proof. The first order conditions for a competitive equilibrium read:

$$\begin{aligned} 2\left(\mu_Y - \theta'_C\sigma_Y^2\right) + \frac{\partial\overline{V}_C(2\rho')}{\partial\theta'_C} &= 2p \\ \mu_Y - r_E\theta'_E\sigma_Y^2 + \frac{\partial\overline{V}_E(2\rho')}{\partial\theta'_E} &= p. \end{aligned}$$

Combining these equations yields:

$$\frac{\partial\overline{V}_E(2\rho')}{\partial\theta'_E} - \frac{1}{2}\frac{\partial\overline{V}_C(2\rho')}{\partial\theta'_C} = \sigma_Y^2\theta'_C\left(\frac{1}{\rho'} - 1\right). \tag{5.4}$$

Full information transmission is achieved at $\rho' = \theta'_C/(r_E\theta'_E) = 1/2$. Market clearing requires $\theta'_E + \theta'_C = 1$. Putting these conditions together and solving yields: $\theta'_E = 2/(2+r_E)$ and $\theta'_C = r_E/(2+r_E)$. Substituting into (5.4) yields the following necessary condition for a competitive equilibrium with full information transmission:

$$\frac{\partial\overline{V}_E(1)}{\partial\theta'_E} - \frac{1}{2}\frac{\partial\overline{V}_C(1)}{\partial\theta'_C} = \sigma_Y^2\frac{r_E}{(2+r_E)}.$$

Since the left-hand side is bounded above, this equation cannot hold for σ_Y sufficiently large. □

The intuition for this negative result is that now there is a tradeoff between full communication which requires $\rho' = 1/2$, and full risk-sharing which requires $\rho' = 1$. The latter motive is more salient when σ_Y^2 is larger. The divergence between motives is due to the fact that by purchasing shares the controlling investor acquires more-diversified Y-risk compared to the expert investor.

5.4. Trading With Private Information

In this section we explore the setting where the expert investor trades shares *after* receiving his private information. There are: a controlling investor who holds exactly θ_C (entrenched investor); an expert investor who is endowed with t^* shares; and many noise traders with a demand that is infinitely elastic at price p. After learning ω, the expert may acquire shares at a constant price p. The expert's final position after trading is denoted by $t(\omega)$. We assume that the expert's strategy $t(\omega)$ is bounded between ε (a very small but positive number) and $1-\theta_C$. We assume that the controlling investor observes $t(\omega)$ before choosing a.[15] The equilibrium construction is inspired by Section 4 in Kartik (2007), but our setting features significant technical challenges requiring a specific analysis. Online Appendix C provides the analysis.

We show that for low values of ω the equilibrium strategy $t(\omega)$ is monotonically increasing and thus fully separating, meaning that the controlling shareholder will be able to infer ω from $t(\omega)$ and so select the best possible a for herself. In this fully separating region the equilibrium strategy solves the following differential equation:

$$t'(\omega) = -\frac{\left(\frac{r_E}{\theta_C}t(\omega)-1\right)\omega}{\left(\frac{r_E}{\theta_C}t(\omega)-1\right)\omega^2+\frac{\theta_C}{\sigma_X^2}p}t(\omega).$$

For high values of ω the expert may choose to "max out" on shares at $t(\omega)=1-\theta_C$ and then rely on cheap talk to convey additional information to the controlling investor. In this pooling region the cheap talk equilibrium resembles that in Section 3. By studying the equilibrium strategy as a function of p and r_E, we show that the expert investor will acquire more shares, and thus will more often have to rely on cheap talk, if the share price is low or if he is less risk-averse. A noteworthy feature of this equilibrium is that while the signaling motive leads to an upward distortion in the amount of shares purchased by the expert, yet the expert never demands so many shares that $\theta_C/r_E t(\omega)=\rho<1$.

[15] We assume that the trading price is not observed by the controlling investor. In a more general model where price is increasing in $t(\omega)$ this assumption may be limiting, to the extent that the equilibrium price reveals additional information about ω beyond the quantity traded by the informed investor.

5.5. Shares Are Allocated As Compensation in a Principal-Agent Relationship

In a principal–agent setting, the principal, not the market, sets the incentive scheme. We consider an expert shareholder (a manager, say) whose compensation contract is designed by the controlling shareholder (the principal). We now show that, if the space of contracts includes a fixed salary plus shares in the enterprise, the optimal compensation scheme assigns the manager too much equity for her to be willing to communicate truthfully with the principal. Thus, in general the principal will face a trade-off between providing the agent with incentives to exert optimal effort and incentives to communicate fully.

The setting is as follows. An agent, who is indexed by E, has private information about an investment and transmits this information to a principal. For simplicity we set $Z_j = 0$ for both principal and agent. The agent's payoff function is as that of the expert investor in the previous sections, with an added moral hazard component: the agent exerts effort e at a private cost $c(e)$. Effort e produces output e. If compensated with a salary S and shares $\theta_E \in [0,1]$, the agent's payoff (net of the portion due to Y which has no strategic effect) is:

$$u_E(e;\theta_E,S) = \underbrace{\overline{V}_E\left(\frac{1-\theta_E}{r_E\theta_E}\right)}_{\text{old preferences}} + \underbrace{\theta_E e - c(e)}_{\text{moral hazard part}} + S.$$

Output is owned by a principal, who is indexed by C and owns the remaining shares $(1-\theta_E)$. After communicating with the agent, the principal chooses an investment strategy and has the payoff function of the controlling investor in the previous sections; in addition, the principal owns the output e. The principal chooses a compensation scheme for the agent comprised of a fixed salary S plus a share θ_E in the enterprise. The principal's payoff net of the portion due to Y is then:

$$u_C(e;\theta_E,S) = \underbrace{\overline{V}_C\left(\frac{1-\theta_E}{r_E\theta_E}\right)}_{\text{old preferences}} + \underbrace{(1-\theta_E)\,e}_{\text{retained output}} - S.$$

Proposition 7 (Trade-off between moral hazard and imperfect communication)**.** *The principal's choice of compensation scheme (salary S plus share θ_E) is such that the agent receives* *no fewer shares than* $\theta_E = 1/(1 + r_E)$, *the amount that he would receive absent the moral hazard problem.*

Proof. The argument is standard in the principal–agent literature. The principal's problem is:

$$\begin{aligned} &\max_{\theta_E, S} && u_C\left(e^*\left(\theta_E\right); \theta_E, S\right) \\ &\text{subject to} \quad : && e^*\left(\theta_E\right) \in \arg\max u_E\left(e; \theta_E, S\right) \ \text{(incentive compatibility of effort)} \\ & && u_E\left(e^*\left(\theta_E\right); \theta_E, S\right) \geq 0 \ \text{(individual rationality).} \end{aligned}$$

By standard arguments, the principal's problem can be re-written as

$$\max_{\theta_E} \overline{V}_C\left(\frac{1-\theta_E}{r_E \theta_E}\right) + \overline{V}_E\left(\frac{1-\theta_E}{r_E \theta_E}\right) + e^*\left(\theta_E\right) - c\left(e^*\left(\theta_E\right)\right).$$

Next, note that $e^*\left(\theta_E\right) - c\left(e^*\left(\theta_E\right)\right)$ is an increasing function of θ_E. To see this, differentiate

$$\begin{aligned} & \frac{\partial}{\partial \theta_E}\left[e^*\left(\theta_E\right) - c\left(e^*\left(\theta_E\right)\right)\right] = \frac{\partial}{\partial \theta_E}\left[\left(1-\theta_E\right) e^*\left(\theta_E\right)\right] + \frac{\partial}{\partial \theta_E}\left[\theta_E e^*\left(\theta_E\right) - c\left(e^*\left(\theta_E\right)\right)\right] \\ = \ & \frac{\partial}{\partial \theta_E}\left[\left(1-\theta_E\right) e^*\left(\theta_E\right)\right] + e^*\left(\theta_E\right) = -e^*\left(\theta_E\right) + \left(1-\theta_E\right) e^{*\prime}\left(\theta_E\right) + e^*\left(\theta_E\right) \\ = \ & \left(1-\theta_E\right) e^{*\prime}\left(\theta_E\right) > 0. \end{aligned}$$

Therefore, the principal's problem is the sum of two functions that both peak when $\frac{1-\theta_E}{r_E \theta_E} = 1$, that is, when $\theta_E = 1/(1 + r_E)$, and of a third function that is increasing in θ_E. □

The optimal contract trades off providing the agent with incentives to exert optimal effort with providing incentives to communicate fully. As a result the agent is assigned too much equity to efficiently communicate with the principal, but less equity than would be optimal if the only incentive issue was effort provision. In equilibrium the agent will feel that the principal's investment strategy is too aggressive and this will lead the agent to misreport his signal. This perspective is different from (though not contradictory of) the take of most of the literature on corporate finance, which is that the CEO might take excessive risk relative to the board's preferences. The difference is due to the fact that in our setting the CEO merely advises, and the board chooses the strategy, so that any predictable bias on the CEO's part will be undone by the board.

6. Conclusion

This paper has provided a theoretical model of information transmission about a risky investment prospect. The model innovates in two ways relative to the cheap-talk literature: first, in that the incentives to communicate are shaped by equity ownership; second, in that equity ownership is determined endogenously, hence the sender and receiver make choices before a cheap-talk game.

The theoretical analysis has been applied to two settings. In the first setting, equity ownership is determined by investors buying shares on a competitive equity market. We have provided conditions under which share-trading delivers perfect communication and full risk-sharing. When frictions lead to a suboptimal outcome, we have identified a novel source of inefficiency: a competitive market for shares fails to reward the positive externality that an expert investor provides to other investors by purchasing shares. The second setting is a principal-agent relationship where equity is granted as compensation by a principal (board) to an agent (CEO). The model has highlighted a novel trade-off between incentivizing effort provision and promoting information transmission. These applications illustrate a conceptual question: how different share allocation mechanisms shape the sender and receiver's choices before a cheap-talk game with direct implications for their alignment of interest.

A limitation must be acknowledged: we assumed that the risk aversion parameters are common knowledge across agents.

Much of the literature on information transmission in organizations has focused on optimizing the allocation of control, following Dessein's (2002) pioneering insight. This paper suggests that, in organizations where ownership can be easily adjusted, allowing the market to allocate ownership can sometimes help solve the information transmission problem.

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Appendix A. Deferred Proofs

The order-n Chebyshev polynomial of the second kind, $U_n(k)$, is a polynomial function of k defined as the unique solution to the following functional difference equation:

$$U_n(k) - 2kU_{n-1}(k) + U_{n-2}(k) = 0, \tag{A.1}$$

with initial conditions $U_{-1}(k) = 0$, $U_0(k) = 1$. The most common expression for $U_n(k)$, and the one that will best serve our purposes is

$$U_n(k) = \begin{cases} \frac{\sin((n+1)\arccos k)}{\sin(\arccos k)} & \text{if } |k| \leq 1 \\ \frac{\sinh((n+1)\operatorname{arccosh} k)}{\sinh(\operatorname{arccosh} k)} & \text{if } k > 1, \end{cases} \tag{A.2}$$

where $\operatorname{arccosh} k$ is nonnegative (refer to expressions 1.4 and 1.33b in Mason and Handscomb, 2003). Because equation (A.1), which generates the family of Chebyshev polynomials, is the same as equation (3.1), there must be a strong connection between the equilibrium of our cheap-talk game and the family $U_n(k)$.

Proposition A.1 (Characterization of cheap-talk equilibrium strategies). *Fix a risk-adjusted holdings ratio ρ. Take any equilibrium with partition cutoffs $\{\omega_0, \omega_1, ..., \omega_N\}$, where $\omega_0 = 0$ and $\omega_N = 1$.*

(1) *Upon learning that ω falls into partition (ω_n, ω_{n+1}), the controlling investor's equilibrium action is $a_C^*(\omega_n, \omega_{n+1}|\theta_C) = (\omega_{n+1} + \omega_n)/2\theta_C$.*
(2) *Equilibrium cutoffs ω_n solve difference equation (3.1).*
(3) *Fix an equilibrium with N partitions. Equilibrium partitions have the form $\omega_n = U_{n-1}(k_\rho)/U_{N-1}(k_\rho)$, where U_n is the order-n Chebyshev polynomial of the second kind.*
(4) *If $\rho = 1$ perfect communication is an equilibrium.*
(5) *Suppose $\rho > 1$, that is, the expert investor (sender) has less than the optimal risk-sharing holding relative to the controlling investor (receiver). Then there are equilibria with an arbitrarily large number of partitions.*
(6) *Suppose $0 < \rho < 1$, that is, the expert investor (sender) has more than the optimal risk-sharing holding relative to the controlling investor (receiver). Then the maximal number of partitions consistent with an equilibrium is $\left\lceil \frac{1}{2}\left(\frac{\pi}{\arccos k_\rho} - 1\right)\right\rceil$ which is increasing in ρ and converges to infinity as $\rho \uparrow 1$. No information can be communicated when $0 < \rho < 3/4$.*
(7) *If $\rho < 0$, that is, the expert investor (sender) is a short-seller, then no information can be communicated in equilibrium.*

Proof. **Part 1.** In equilibrium, after receiving a message the controlling investor (receiver) knows that the state ω is distributed uniformly over some interval (ω_n, ω_{n+1}). Then her equilibrium action $a_C^*(\omega_n, \omega_{n+1}|\theta_C)$ is given by

$$a^*(\omega_n, \omega_{n+1}|\theta_C) = \arg\max_a - \int_{\omega_n}^{\omega_{n+1}} (a\theta_C - \omega)^2 \, d\omega.$$

The first-order condition w.r.t. a reads:

$$0 = \int_{\omega_n}^{\omega_{n+1}} (a\theta_C - \omega)\, d\omega = a\theta_C (\omega_{n+1} - \omega_n) - \frac{\omega_{n+1}^2 - \omega_n^2}{2} = a\theta_C - \frac{\omega_{n+1} + \omega_n}{2},$$

hence the desired expression for $a_C^*(\omega_n, \omega_{n+1}|\theta_C)$.

Part 2. At a cutpoint between different intervals, the sender needs to be indifferent between inducing the equilibrium action associated to either interval. Thus if the sender knows that $\omega = \omega_n$ it must be that:

$$\left(\frac{\omega_{n+1} + \omega_n}{2\theta_C}\theta_E r_E - \omega_n\right)^2 = \left(\frac{\omega_n + \omega_{n-1}}{2\theta_C}\theta_E r_E - \omega_n\right)^2.$$

Taking square roots on both sides of the equation while switching the sign of the right-hand side yields:

$$\frac{\omega_{n+1} + \omega_n}{2\theta_C}\theta_E r_E - \omega_n = -\frac{\omega_n + \omega_{n-1}}{2\theta_C}\theta_E r_E + \omega_n$$

Rearranging we get the following difference equation:

$$\omega_{n+1} = 2\left(2\frac{\theta_C}{r_E\theta_E} - 1\right)\omega_n - \omega_{n-1} \tag{A.3}$$

which, when incremented, reads as in (3.1).

Part 3. For any k_ρ, all cheap talk equilibrium partitions satisfy equation (3.1) with initial condition $\omega_0 = 0$. Chebyshev polynomials $U_n(k_\rho)$ are the unique solutions to (3.1) with initial conditions $U_{-1}(k_\rho) = 0, U_0(k_\rho) = 1$. Thus all cheap talk equilibria must take the form $\{\omega_n(k_\rho)\}_n = \{\alpha U_{n-1}(k_\rho)\}_n$, where $\alpha = U_{N-1}(k_\rho)$ follows from the equilibrium requirement that $\omega_N = 1$.

Part 4. If $\rho = 1$ there is no conflict of interest as discussed in Section 2.6.4, so perfect communication is an equilibrium.

Part 5. $\rho > 1$ is equivalent to $k_\rho > 1$. For any $k > 1$ we have $U_n(k) > U_{n-1}(k)$ because $\operatorname{arccosh} k > 0$ by definition, hence $\omega_n(k_\rho) = \alpha U_{n-1}(k_\rho)$ is monotonic in n. Furthermore, for any N one can choose α such that $\alpha U_{N-1}(k_\rho) = 1$, and thus construct a communication equilibrium with exactly N partitions.

Part 6. $\rho < 1$ is equivalent to $|k| < 1$. In this case the Chebyshev polynomial $U_n(k)$ is not monotonic in n, and so we need to worry about the monotonicity constraint $\omega_n > \omega_{n-1}$ which is required by the equilibrium definition. The monotonicity constraint requires that, for all

$n = 1, ..., N,$

$$0 < sgn\left[\omega_n - \omega_{n-1}\right] = sgn\left[U_{n-1}\left(k_\rho\right) - U_{n-2}\left(k_\rho\right)\right]. \tag{A.4}$$

Let's provide conditions on N such that this is true. Denote $\phi = \arccos k$. Since $|k| < 1$, $\arccos k \in [0, \pi]$ and thus $\sin(\arccos k) \geq 0$. Then from (A.2) we get:

$$sgn\left[U_n(k) - U_{n-1}(k)\right] = sgn\left[\sin((n+1)\phi) - \sin(n\phi)\right].$$

Now,

$$\sin((n+1)\phi) - \sin(n\phi) = \sin\left(n\phi + \frac{\phi}{2} + \frac{\phi}{2}\right) - \sin\left(n\phi + \frac{\phi}{2} - \frac{\phi}{2}\right) = 2\cos\left(n\phi + \frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right). \tag{A.5}$$

where we have used the product-to-sum identity $\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos(\alpha)\sin(\beta)$. Let's start by signing the second term; by the half-angle formula,

$$\sin\left(\frac{\phi}{2}\right) = \sqrt{\frac{1 - \cos(\phi)}{2}} sgn\left(2\pi - \phi + 4\pi\left\lfloor\frac{\phi}{4\pi}\right\rfloor\right).$$

By definition, $\phi = \arccos(k) \in [0, \pi]$, and so the sgn operator returns $+1$. Therefore, the whole right-hand side is positive, which shows that $\sin(\phi/2) > 0$. Thus, expression (A.5) has the same sign as its first term. That term, $2\cos(n\phi + \phi/2)$, is positive as long as:

$$n\phi + \frac{\phi}{2} < \frac{\pi}{2}.$$

Solve for n to get:

$$n < \frac{1}{2}\left(\frac{\pi}{\phi} - 1\right).$$

Thus (A.4) holds if $n - 1 < \frac{1}{2}\left(\frac{\pi}{\arccos k_\rho} - 1\right)$, or $n < \frac{1}{2}\left(\frac{\pi}{\arccos k_\rho} + 1\right)$. Therefore the maximal integer N_ρ that is consistent with (A.4) is:

$$N_\rho = \left\lceil\frac{1}{2}\left(\frac{\pi}{\arccos k_\rho} + 1\right)\right\rceil - 1 = \left\lceil\frac{1}{2}\left(\frac{\pi}{\arccos k_\rho} - 1\right)\right\rceil.$$

This N_ρ is increasing in k_ρ because $arccos(\cdot)$ is a decreasing function over the interval $[-1, 1]$, and has a vertical asymptote at $k_\rho = 1$ because $arccos(1) = 0$. No information can be communicated when $\rho < 3/4$ for in this case $k_\rho < 1/2$ and then $\arccos(k_\rho) > \pi/3$ whence $N_\rho = 1$.

Part 7. If $\rho < 0$ then expert and controlling investors have opposing interests: if an action increases the controlling investor's payoff then it decreases the expert investor's payoff. So no communication can credibly take place in equilibrium. $\square$

Proposition A.2 (Closed-form solutions for controlling investor's new-venture expected payoff)**.** *Fix ρ. Up to the additive constant $\frac{\sigma_X^2}{2}\mathbb{E}\left(\omega^2\right)$, the controlling investor's new-venture expected payoffs are as follows:*

(1) *If $\rho < 3/4$, the controlling investor's equilibrium payoff equals $-\sigma_X^2/24$.*
(2) *If $3/4 \leq \rho \leq 1$, the controlling investor's payoff in an equilibrium with $N > 1$ partitions is*

$$\begin{aligned} V_C(\rho, N) &= \frac{\sigma_X^2}{6}\frac{\rho-1}{(4\rho-1)}\left(\frac{3\rho}{\sin^2(N \arccos k_\rho)} - 1\right) \\ &\leq \overline{V}_C(\rho) = \frac{\sigma_X^2}{6}\frac{(\rho-1)(3\rho-1)}{(4\rho-1)}. \end{aligned} \tag{A.6}$$

The payoff $V_C(\rho, N)$ is increasing in N. For any ρ, there exists a value $\overline{\rho} \in [\rho, 1)$ such that $V_C(\overline{\rho}, N_{\overline{\rho}})$ attains the bound $\overline{V}_C(\overline{\rho})$.

(3) *If $\rho \geq 1$, the controlling investor's payoff in an equilibrium with $N > 1$ partitions is*

$$\begin{aligned} V_C(\rho, N) &= \frac{\sigma_X^2}{6}\frac{1-\rho}{4\rho-1}\left(\frac{3\rho}{\sinh^2(N \operatorname{arccosh} k_\rho)} + 1\right) \\ &\leq \overline{V}_C(\rho) = \frac{\sigma_X^2}{6}\frac{1-\rho}{(4\rho-1)}. \end{aligned}$$

The payoff $V_C(\rho, N)$ is increasing in N and $\lim_{N\to\infty} V_C(\rho, N) = \overline{V}_C(\rho)$.

Proof. In light of Lemma B.1, the investor's expected utility in an equilibrium when N intervals are being communicated is:

$$\begin{aligned} V_C(\rho, N) &= -\frac{\sigma_X^2}{2}\sum_{n=0}^{N-1}\int_{\omega_n}^{\omega_{n+1}} \left(\omega - a_C^*(\omega_n, \omega_{n+1}|\theta_C)\,\theta_C\right)^2 \, \mathrm{d}\omega \\ &= -\frac{\sigma_X^2}{6}\sum_{n=0}^{N-1}\left[\left(\omega_{n+1} - a_C^*(\omega_n, \omega_{n+1}|\theta_C)\,\theta_C\right)^3 - \left(\omega_n - a_C^*(\omega_n, \omega_{n+1}|\theta_C)\,\theta_C\right)^3\right] \\ &= -\frac{\sigma_X^2}{6}\sum_{n=0}^{N-1}\left[\left(\omega_{n+1} - \frac{\omega_{n+1}+\omega_n}{2}\right)^3 - \left(\omega_n - \frac{\omega_{n+1}+\omega_n}{2}\right)^3\right] \\ &= -\frac{\sigma_X^2}{6}\sum_{n=0}^{N-1}\left[\frac{1}{4}\left(\omega_{n+1}-\omega_n\right)^3\right] = -\frac{\sigma_X^2}{24}\frac{1}{\left[U_{N-1}(k_\rho)\right]^3}\sum_{n=0}^{N-1}\left[U_n(k_\rho) - U_{n-1}(k_\rho)\right]^3, \end{aligned} \tag{A.7}$$

where the third and the fifth equality obtain from Proposition A.1 parts 1 and 3, respectively. Now we establish the following preliminary result:

$$\sum_{n=0}^{N-1}\left[U_n(k) - U_{n-1}(k)\right]^3 = \begin{cases} -\frac{1-k}{2k+1}\left(2 - 3\frac{1+k}{\sin^2(N\arccos k)}\right)\left(U_{N-1}(k)\right)^3 & \text{if } |k| \leq 1 \\ -\frac{1-k}{2k+1}\left(2 + 3\frac{1+k}{\sinh^2(N \operatorname{arccosh} k)}\right)\left(U_{N-1}(k)\right)^3 & \text{if } k \geq 1. \end{cases} \tag{A.8}$$

Let's start by noting that, using Euler's formula, we have:

$$\sin n\phi = \frac{e^{n\phi\iota} - e^{-n\phi\iota}}{2\iota} \text{ and } \sinh n\phi = \frac{e^{n\phi} - e^{-n\phi}}{2},$$

thus we may write:

$$U_{n-1}(k) = \frac{e^{n\phi\iota^{\mathbf{1}_{[k<1]}}} - e^{-n\phi\iota^{\mathbf{1}_{[k<1]}}}}{e^{\phi\iota^{\mathbf{1}_{[k<1]}}} - e^{-\phi\iota^{\mathbf{1}_{[k<1]}}}} = \frac{e^{\xi n\phi} - e^{-\xi n\phi}}{e^{\xi\phi} - e^{-\xi\phi}}, \tag{A.9}$$

where we denote $\xi := \iota^{\mathbf{1}_{[k<1]}}$ and, with a slight abuse of notation, $\phi = \arccos k$ for $|k| \leq 1$ and $\phi = \operatorname{arccosh} k$ for $k > 1$. Then (note the inversion in the indices):

$$\begin{aligned}
& \sum_{n=0}^{N-1} \left(U_{n-1}(k) - U_n(k)\right)^3 \\
= \; & \sum_{n=0}^{N-1} \left(\frac{e^{\xi n\phi} - e^{-\xi n\phi}}{e^{\xi\phi} - e^{-\xi\phi}} - \frac{e^{\xi(n+1)\phi} - e^{-\xi(n+1)\phi}}{e^{\xi\phi} - e^{-\xi\phi}} \right)^3 \\
= \; & \sum_{n=0}^{N-1} \left(\frac{1}{e^{\xi\phi} - e^{-\xi\phi}} \right)^3 \left(e^{\xi n\phi} \left(1 - e^{\xi\phi}\right) + e^{-\xi(n+1)\phi} \left(1 - e^{\xi\phi}\right) \right)^3 \\
= \; & \left(\frac{1 - e^{\xi\phi}}{e^{\xi\phi} - e^{-\xi\phi}} \right)^3 \sum_{n=0}^{N-1} \left(e^{\xi n\phi} + e^{-\xi(n+1)\phi} \right)^3 \\
= \; & \left(\frac{1 - e^{\xi\phi}}{e^{\xi\phi} - e^{-\xi\phi}} \right)^3 \sum_{n=0}^{N-1} \left(e^{3\xi n\phi} + 3e^{\xi(n-1)\phi} + 3e^{-\xi(n+2)\phi} + e^{-3\xi(n+1)\phi} \right) \\
= \; & \left(\frac{1 - e^{\xi\phi}}{e^{\xi\phi} - e^{-\xi\phi}} \right)^3 \left[\frac{e^{3\xi N\phi} - 1}{e^{3\xi\phi} - 1} + 3\frac{e^{\xi(N-1)\phi} - e^{-\xi\phi}}{e^{\xi\phi} - 1} + 3\frac{e^{-\xi(N+2)\phi} - e^{-2\xi\phi}}{e^{-\xi\phi} - 1} + \frac{e^{-3\xi(N+1)\phi} - e^{-3\xi\phi}}{e^{-3\xi\phi} - 1} \right] \\
= \; & \left(\frac{1 - e^{\xi\phi}}{e^{\xi\phi} - e^{-\xi\phi}} \right)^3 \left[\frac{e^{3\xi N\phi} - 1 + 1 - e^{-3\xi N\phi}}{e^{3\xi\phi} - 1} + 3\frac{e^{\xi(N-1)\phi} - e^{-\xi\phi} + e^{-\xi\phi} - e^{-\xi(N+1)\phi}}{e^{\xi\phi} - 1} \right] \\
= \; & \left(\frac{1 - e^{\xi\phi}}{e^{\xi\phi} - e^{-\xi\phi}} \right)^3 \left[\frac{e^{3\xi N\phi} - e^{-3\xi N\phi}}{e^{3\xi\phi} - 1} + 3\frac{e^{\xi(N-1)\phi} - e^{-\xi(N+1)\phi}}{e^{\xi\phi} - 1} \right].
\end{aligned}$$

Now let's break down the above expression into two addends, starting with:

$$\begin{aligned}
& \left(\frac{1 - e^{\xi\phi}}{e^{\xi\phi} - e^{-\xi\phi}} \right)^3 3\frac{e^{\xi(N-1)\phi} - e^{-\xi(N+1)\phi}}{e^{\xi\phi} - 1} \\
= \; & -3\frac{\left(1 - e^{\xi\phi}\right)^2}{\left(e^{\xi\phi} - e^{-\xi\phi}\right)^3} \left(e^{\xi(N-1)\phi} - e^{-\xi(N+1)\phi} \right) = -3\frac{\left(1 - e^{\xi\phi}\right)^2 e^{-\xi\phi}}{\left(e^{\xi\phi} - e^{-\xi\phi}\right)^2} \frac{e^{\xi N\phi} - e^{-\xi N\phi}}{e^{\xi\phi} - e^{-\xi\phi}} \\
= \; & -3\frac{\left(e^{\xi\phi} + e^{-\xi\phi} - 2\right)}{\left(e^{\xi\phi} - e^{-\xi\phi}\right)^2} U_{N-1}(k) = -6\frac{(k-1)}{\left(e^{\xi\phi} - e^{-\xi\phi}\right)^2} U_{N-1}(k),
\end{aligned}$$

where the last line follows from

$$e^{\xi\phi} + e^{-\xi\phi} = 2k, \tag{A.10}$$

which is true because $2\cos\phi = e^{\iota\phi} + e^{-\iota\phi}$ and $2\cosh\phi = e^{\phi} + e^{-\phi}$. The next addend reads:

$$\begin{aligned}
& \left(\frac{1-e^{\xi\phi}}{e^{\xi\phi}-e^{-\xi\phi}}\right)^3 \frac{e^{3\xi N\phi}-e^{-3\xi N\phi}}{e^{3\xi\phi}-1} \\
= \quad & \frac{1-3e^{\xi\phi}+3e^{2\xi\phi}-e^{3\xi\phi}}{\left(e^{3\xi\phi}-1\right)\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} \cdot \frac{\left(e^{\xi N\phi}-e^{-\xi N\phi}\right)\left(e^{2\xi N\phi}+e^{-2\xi N\phi}+1\right)}{e^{\xi\phi}-e^{-\xi\phi}} \\
= \quad & \frac{1-e^{3\xi\phi}-3e^{\xi\phi}\left(1-e^{\xi\phi}\right)}{\left(e^{3\xi\phi}-1\right)} \cdot \frac{U_{N-1}(k)\left(e^{2\xi N\phi}-2+e^{-2\xi N\phi}+3\right)}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} \\
= \quad & \left(-1+\frac{3e^{\xi\phi}\left(1-e^{\xi\phi}\right)}{\left(1-e^{\xi\phi}\right)\left(1+e^{\xi\phi}+e^{2\xi\phi}\right)}\right) U_{N-1}(k)\, \frac{\left(e^{\xi N\phi}-e^{-\xi N\phi}\right)^2+3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} \\
= \quad & \left(-\frac{1-2e^{\xi\phi}+e^{2\xi\phi}}{1+e^{\xi\phi}+e^{2\xi\phi}}\right) U_{N-1}(k) \left(\left(U_{N-1}(k)\right)^2+\frac{3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right) \\
= \quad & \left(-\frac{e^{-\xi\phi}-2+e^{\xi\phi}}{e^{-\xi\phi}+1+e^{\xi\phi}}\right) U_{N-1}(k) \left(\left(U_{N-1}(k)\right)^2+\frac{3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right) \\
= \quad & \frac{2-2k}{2k+1} U_{N-1}(k) \left(\left(U_{N-1}(k)\right)^2+\frac{3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right),
\end{aligned}$$

where the last line follows from equation (A.10). Putting both addends together:

$$\begin{aligned}
& \sum_{n=0}^{N-1} \left(U_{n-1}(k) - U_n(k)\right)^3 \\
= \quad & \frac{2-2k}{2k+1} U_{N-1}(k) \left(\left(U_{N-1}(k)\right)^2+\frac{3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right) - 6\frac{(k-1)}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} U_{N-1}(k) \\
= \quad & U_{N-1}(k) \left(\frac{1-k}{k+1/2}\left(\left(U_{N-1}(k)\right)^2+\frac{3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right)+6\frac{(1-k)}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right) \\
= \quad & \frac{1-k}{k+1/2}\left(1+\frac{3}{\left(U_{N-1}(k)\right)^2\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}+6\frac{k+1/2}{\left(U_{N-1}(k)\right)^2\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right)\left(U_{N-1}(k)\right)^3 \\
= \quad & \frac{1-k}{k+1/2}\left(1+6\frac{k+1}{\left(U_{N-1}(k)\right)^2\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right)\left(U_{N-1}(k)\right)^3 \\
= \quad & \frac{1-k}{k+1/2}\left(1+6\frac{1+k}{\left(e^{\xi N\phi}-e^{-\xi N\phi}\right)^2}\right)\left(U_{N-1}(k)\right)^3.
\end{aligned}$$

Expression (A.8) follows after noting that:

$$\left(e^{\xi N\phi}-e^{-\xi N\phi}\right)^2 = \begin{cases} \left(2\iota\sin N\phi\right)^2 = -4\sin^2 N\phi & \text{if } |k| \le 1 \\ \left(2\sinh N\phi\right)^2 = 4\sinh^2 N\phi & \text{if } k \ge 1. \end{cases} \tag{A.11}$$

Let us now turn to proving the proposition's statement.

(1) If $\rho < 3/4$, no information can be communicated. In this case the investor would choose action $1/2\theta_C$ (see Proposition A.1) giving rise to an expected payoff which, by (2.2), equals

$$-\frac{\sigma_X^2}{2}\mathbb{E}\left(\frac{1}{2}-\omega\right)^2 = -\frac{\sigma_X^2}{24}.$$

(2) In the case $3/4 \leq \rho \leq 1$, we have $|k_\rho| \leq 1$ and substituting the relevant expression from expression (A.8) into (A.7) we get

$$V_C(\rho, N) = \frac{\sigma_X^2}{24}\frac{1-k_\rho}{2k_\rho+1}\left(2-3\frac{1+k_\rho}{\sin^2(N\arccos k_\rho)}\right) = \frac{\sigma_X^2}{6}\frac{\rho-1}{4\rho-1}\left(3\frac{\rho}{\sin^2(N\arccos k_\rho)}-1\right),$$

where the last equality follows from substituting $2\rho-1$ for k_ρ and collecting terms. To check that $V_C(\rho, N)$ is increasing in N, write:

$$V_C(\rho, N) - V_C(\rho, N-1) = \frac{\sigma_X^2}{6}\frac{\rho-1}{4\rho-1}3\rho\left(\frac{1}{\sin^2(N\arccos k_\rho)} - \frac{1}{\sin^2((N-1)\arccos k_\rho)}\right).$$

Since $\rho > 3/4$ the factor multiplying the parenthesis is negative and thus $V_C(\rho, N) \geq V_C(\rho, N-1)$ if and only if:

$$\sin^2(N\arccos k_\rho) \geq \sin^2((N-1)\arccos k_\rho). \tag{A.12}$$

Let $\phi = \arccos k_\rho$, and note that by assumption

$$N \leq N_\rho = \left\lceil \frac{1}{2}\left(\frac{\pi}{\phi}-1\right)\right\rceil = \left\lceil \frac{1}{2}\left(\frac{\pi}{\phi}+1\right)\right\rceil - 1 \leq \frac{1}{2}\left(\frac{\pi}{\phi}+1\right),$$

whence $(2N-1)\phi < \pi$ which implies $N\phi < \pi$. Then $\sin(N\phi) > 0$ and so (A.12) is equivalent to:

$$\sin(N\phi) \geq \sin((N-1)\phi).$$

Now:

$$\begin{aligned} \sin(N\phi) - \sin((N-1)\phi) &= \sin\left(N\phi - \frac{\phi}{2} + \frac{\phi}{2}\right) - \sin\left(N\phi - \frac{\phi}{2} - \frac{\phi}{2}\right) \\ &= 2\cos\left(N\phi - \frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right), \end{aligned}$$

and since $\sin(\phi/2) > 0$ we have that $\sin(N\phi) \geq \sin((N-1)\phi)$ if and only if $\cos(N\phi - \phi/2)$, is positive, i.e., if and only if

$$N\phi - \frac{\phi}{2} \leq \frac{\pi}{2},$$

or after solving for N, if

$$N \leq \frac{1}{2}\left(\frac{\pi}{\phi}+1\right),$$

which was already shown above to be true. To establish the tightness of the upper bound, note that

$$V_C(\overline{\rho}, N_{\overline{\rho}}) = \frac{\sigma_X^2}{6}\frac{\overline{\rho}-1}{(4\overline{\rho}-1)}\left(\frac{3\overline{\rho}}{\sin^2(N_{\overline{\rho}}\arccos k_{\overline{\rho}})}-1\right).$$

Now, the approximation is tight whenever

$$\left\lceil \frac{1}{2}\left(\frac{\pi}{\arccos k_{\overline{\rho}}}-1\right)\right\rceil \arccos k_{\overline{\rho}} = \frac{\pi}{2},$$

or

$$\left\lceil \frac{1}{2}\frac{\pi}{\arccos k_{\overline{\rho}}}-\frac{1}{2}\right\rceil = \frac{1}{2}\frac{\pi}{\arccos k_{\overline{\rho}}},$$

which holds whenever $\frac{1}{2}\frac{\pi}{\arccos k_{\overline{\rho}}} \in \mathbb{Z}$ and this occurs arbitrarily often as $\lim_{\overline{\rho}\to 1}\arccos k_{\overline{\rho}} = 0$.

(3) In the case $\rho \geq 1$ we have $|k_\rho| \geq 1$, and substituting the relevant expression from expression (A.8) into (A.7) we get

$$\begin{aligned} V_C(\rho, N) &= \frac{\sigma_X^2}{24}\frac{1-k_\rho}{2k_\rho+1}\left(2+3\frac{1+k_\rho}{\sinh^2(N\operatorname{arccosh} k_\rho)}\right) \\ &= \frac{\sigma_X^2}{6}\frac{1-\rho}{4\rho-1}\left(1+3\frac{\rho}{\sinh^2(N\operatorname{arccosh} k_\rho)}\right). \end{aligned}$$

This expression is negative and is increasing in N because the function $\sinh(\cdot)$ is monotonically increasing. As $N \to \infty$ this converges to $\frac{\sigma_X^2}{6}\frac{1-\rho}{(4\rho-1)}$.

□

The order-n Chebyshev polynomial of the first kind, $T_n(x)$, is a polynomial function of x defined as the unique solution to the functional difference equation (A.1) with initial conditions $T_0(x) \equiv 1, T_1(x) \equiv x$. Chebyshev polynomial of the first kind admit the following representation:

$$T_n(k) = \begin{cases} \cos(n\arccos k) & \text{if } |k| \leq 1 \\ \cosh(n\operatorname{arccosh} k) & \text{if } k > 1. \end{cases}$$

Proposition A.3 (Closed-form solutions for expert investor's new-venture expected payoff)**.** *Fix ρ. Up to the additive constant $\frac{\sigma_X^2}{2r_E}\mathbb{E}(\omega^2)$, the expert investor's new venture's expected payoffs are as follows:*

(1) *If $\rho < 3/4$, no information can be communicated. In this case the expert investor's equilibrium payoff equals*

$$-\frac{\sigma_X^2}{2r_E}\mathbb{E}\left(\frac{1}{2\rho}-\omega\right)^2.$$

(2) *If $3/4 \leq \rho \leq 1$, the expert investor's payoff in an equilibrium with $N > 1$ partitions is:*

$$\begin{aligned} V_E(\rho, N) &= -\frac{\sigma_X^2}{6r_E}\frac{1-\rho}{\rho(4\rho-1)}\left(3\frac{2\rho-1}{\sin^2(N\arccos k_\rho)}+3-4\rho\right) \\ &\leq \overline{V}_E(\rho) = -\frac{\sigma_X^2}{3r_E}\frac{1-\rho}{(4\rho-1)} \end{aligned} \quad (A.13)$$

The payoff $V_E(\rho, N)$ is increasing in N. For any ρ, there exists a value $\overline{\rho} \in [\rho, 1)$ such that $V_E(\overline{\rho}, N_{\overline{\rho}})$ attains the bound $\overline{V}_E(\overline{\rho})$.

(3) *If $\rho \geq 1$, the expert investor's payoff in an equilibrium with $N > 1$ partitions is:*

$$\begin{aligned} V_E(\rho, N) &= -\frac{\sigma_X^2}{6r_E}\frac{(\rho-1)}{\rho(4\rho-1)}\left(3\frac{2\rho-1}{\sinh^2(N\operatorname{arccosh} k_\rho)}+4\rho-3\right) \\ &\leq \overline{V}_E(\rho) = -\frac{\sigma_X^2}{6r_E}\frac{\rho-1}{\rho(4\rho-1)}(4\rho-3), \end{aligned}$$

The payoff $V_E(\rho, N)$ is increasing in N and $\lim_{N\to\infty} V_E(\rho, N) = \overline{V}_E(\rho)$.

Proof. In light of Lemma B.1, the expert investor's expected utility in an equilibrium when N intervals are being communicated is

$$\begin{aligned} V_E(\rho, N) &= -\frac{\sigma_X^2}{2r_E}\sum_{n=0}^{N-1}\int_{\omega_n}^{\omega_{n+1}}(\omega-\theta_E r_E a^*(\omega_n,\omega_{n+1}|\theta_C))^2\,\mathrm{d}\omega \\ &= -\frac{\sigma_X^2}{6r_E}\sum_{n=0}^{N-1}\left[\left(\omega_{n+1}-\theta_E r_E\frac{\omega_{n+1}+\omega_n}{2\theta_C}\right)^3-\left(\omega_n-\theta_E r_E\frac{\omega_{n+1}+\omega_n}{2\theta_C}\right)^3\right] \\ &= -\frac{\sigma_X^2}{6r_E}\sum_{n=0}^{N-1}\left[\left(\frac{U_n(k_\rho)}{U_{N-1}(k_\rho)}-\frac{U_n(k_\rho)+U_{n-1}(k_\rho)}{(k_\rho+1)U_{N-1}(k_\rho)}\right)^3-\left(\frac{U_{n-1}(k_\rho)}{U_{N-1}(k_\rho)}-\frac{U_n(k_\rho)+U_{n-1}(k_\rho)}{(k_\rho+1)U_{N-1}(k_\rho)}\right)^3\right] \\ &= -\frac{\sigma_X^2}{6r_E(k_\rho+1)^3U_{N-1}(k_\rho)^3}\sum_{n=0}^{N-1}\left[(k_\rho U_n(k_\rho)-U_{n-1}(k_\rho))^3-(k_\rho U_{n-1}(k_\rho)-U_n(k_\rho))^3\right] \\ &= \frac{-\sigma_X^2}{6r_E(k_\rho+1)^3U_{N-1}(k_\rho)^3}\sum_{n=0}^{N-1}\left[T_{n+1}(k_\rho)^3+T_n(k_\rho)^3\right], \end{aligned} \quad (A.14)$$

where the last line follows by an identity linking the Chebyshev polynomials of the first and second kinds, $T_n(k_\rho) = U_n(k_\rho) - k_\rho U_{n-1}(k_\rho)$, and by incrementing this identity and applying the defining linear difference equation

$$T_{n+1}(k_\rho) = U_{n+1}(k_\rho) - k_\rho U_n(k_\rho) = 2k_\rho U_n(k_\rho) - U_{n-1}(k_\rho) - k_\rho U_n(k_\rho) = k_\rho U_n(k_\rho) - U_{n-1}(k_\rho).$$

Now we establish the following preliminary result:

$$\sum_{n=0}^{N-1}\left(T_{n+1}\left(k\right)\right)^{3}+\left(T_{n}\left(k\right)\right)^{3}=\begin{cases}\frac{(k+1)^{2}(k-1)}{2k+1}\left(2k-1-3\frac{k}{\sin^{2}(N\arccos k)}\right)\left(U_{N-1}\left(k\right)\right)^{3} & \text{if } |k|\leq 1\\ \frac{(k+1)^{2}(k-1)}{2k+1}\left(2k-1+3\frac{k}{\sinh^{2}(N\operatorname{arccosh} k)}\right)\left(U_{N-1}\left(k\right)\right)^{3} & \text{if } k\geq 1.\end{cases} \tag{A.15}$$

Let's start from the product-to-sum identity for Chebyshev polynomials $2T_{I}\left(k\right)T_{n}\left(k\right)=T_{I+n}+T_{|I-n|}$ which gives:

$$\begin{aligned}\left(T_{n}\left(k\right)\right)^{3} &= T_{n}\left(k\right)\left(T_{n}\left(k\right)\right)^{2}=T_{n}\left(k\right)\left(\frac{T_{2n}\left(k\right)+1}{2}\right)=\frac{1}{2}T_{2n}\left(k\right)T_{n}\left(k\right)+\frac{1}{2}T_{n}\left(k\right)\\ &= \frac{1}{4}\left(T_{3n}\left(k\right)+T_{n}\left(k\right)\right)+\frac{1}{2}T_{n}\left(k\right)=\frac{1}{4}T_{3n}\left(k\right)+\frac{3}{4}T_{n}\left(k\right).\end{aligned}$$

Hence:

$$\sum_{n=0}^{N-1}\left(T_{n+1}\left(k\right)\right)^{3}+\left(T_{n}\left(k\right)\right)^{3}=\frac{1}{4}\sum_{n=0}^{N-1}T_{3n+3}\left(k\right)+3T_{n+1}\left(k\right)+T_{3n}\left(k\right)+3T_{n}\left(k\right).$$

By Euler's formula:

$$\cos n\phi=\frac{e^{\iota n\phi}+e^{-\iota n\phi}}{2}\quad\text{and}\quad\cosh n\phi=\frac{e^{n\phi}+e^{-n\phi}}{2},$$

and so Chebyshev polynomials of the first kind may be written as:

$$\begin{aligned}T_{n}\left(k\right) &= \frac{e^{\iota^{\mathbf{1}[k<1]}n\phi}+e^{-\iota^{\mathbf{1}[k<1]}n\phi}}{2}\\ &= \frac{e^{\xi n\phi}+e^{-\xi n\phi}}{2},\end{aligned}$$

where ξ and ϕ are defined in the proof of Proposition A.2. Then:

$$\begin{aligned}&\sum_{n=0}^{N-1}T_{n+1}\left(k\right)^{3}+T_{n}\left(k\right)^{3}\\ =&\ \frac{1}{8}\sum_{n=0}^{N-1}\left\{\begin{array}{c}e^{\xi(3n+3)\phi}+e^{-\xi(3n+3)\phi}+3e^{\xi(n+1)\phi}+3e^{-\xi(n+1)\phi}\\ +e^{\xi 3n\phi}+e^{-\xi 3n\phi}+3e^{\xi n\phi}+3e^{-\xi n\phi}\end{array}\right\}\\ =&\ \frac{1}{8}\sum_{n=0}^{N-1}\left\{\begin{array}{l}e^{\xi 3n\phi}\left(e^{\xi 3\phi}+1\right)+e^{-\xi 3n\phi}\left(e^{-\xi 3\phi}+1\right)\\ +3e^{\xi n\phi}\left(e^{\xi\phi}+1\right)+3e^{-\xi n\phi}\left(e^{-\xi\phi}+1\right)\end{array}\right\}\\ =&\ \frac{1}{8}\left(\begin{array}{l}\frac{e^{\xi 3\phi N_{\rho}}-1}{e^{\xi 3\phi}-1}\left(e^{\xi 3\phi}+1\right)+\frac{e^{-\xi 3\phi N_{\rho}}-1}{e^{-\xi 3\phi}-1}\left(e^{-\xi 3\phi}+1\right)\\ +3\frac{e^{\xi\phi N_{\rho}}-1}{e^{\xi\phi}-1}\left(e^{\xi\phi}+1\right)+3\frac{e^{-\xi\phi N_{\rho}}-1}{e^{-\xi\phi}-1}\left(e^{-\xi\phi}+1\right)\end{array}\right).\end{aligned}$$

Let's look at these in turn:

$$3\frac{e^{\xi\phi N}-1}{e^{\xi\phi}-1}\left(e^{\xi\phi}+1\right)+3\frac{e^{-\xi\phi N}-1}{e^{-\xi\phi}-1}\left(e^{-\xi\phi}+1\right)$$

$$
\begin{aligned}
&= -3\frac{\left(e^{\xi\phi N}-1\right)\left(e^{\xi\phi}-e^{-\xi\phi}\right)}{\left(e^{\xi\phi}-1\right)\left(e^{-\xi\phi}-1\right)}+3\frac{\left(e^{-\xi\phi N}-1\right)\left(e^{\xi\phi}-e^{-\xi\phi}\right)}{\left(e^{-\xi\phi}-1\right)\left(e^{\xi\phi}-1\right)}\\
&= -3\frac{\left(e^{\xi\phi}-e^{-\xi\phi}\right)\left(e^{\xi\phi N}-1-e^{-\xi\phi N}+1\right)}{2-e^{\xi\phi}-e^{-\xi\phi}}\\
&= 3\frac{\left(e^{\xi\phi/2}-e^{-\xi\phi/2}\right)\left(e^{\xi\phi/2}+e^{-\xi\phi/2}\right)\left(e^{\xi\phi N}-e^{-\xi\phi N}\right)}{e^{\xi\phi}-2+e^{-\xi\phi}}\\
&= 3\frac{\left(e^{\xi\phi/2}+e^{-\xi\phi/2}\right)^2\left(e^{\xi\phi N}-e^{-\xi\phi N}\right)}{e^{\xi\phi}-e^{-\xi\phi}}\\
&= 3\left(e^{\xi\phi}+e^{-\xi\phi}+2\right)U_{N-1}(k)\\
&= 6(k+1)U_{N-1}(k),
\end{aligned}
$$

where the last line follows from equation (A.10). The other part reads as follows:

$$
\begin{aligned}
&\frac{e^{\xi3\phi N}-1}{e^{\xi3\phi}-1}\left(e^{\xi3\phi}+1\right)+\frac{e^{-\xi3\phi N}-1}{e^{-\xi3\phi}-1}\left(e^{-\xi3\phi}+1\right)\\
=\;& -\frac{\left(e^{\xi3\phi N}-1\right)\left(e^{\xi3\phi}-e^{-\xi3\phi}\right)}{\left(e^{\xi3\phi}-1\right)\left(e^{-\xi3\phi}-1\right)}+\frac{\left(e^{-\xi3\phi N}-1\right)\left(e^{\xi3\phi}-e^{-\xi3\phi}\right)}{\left(e^{-\xi3\phi}-1\right)\left(e^{\xi3\phi}-1\right)}\\
=\;& -\frac{\left(e^{\xi3\phi}-e^{-\xi3\phi}\right)\left(e^{\xi3\phi N}-1-e^{-\xi3\phi N}+1\right)}{2-e^{\xi3\phi}-e^{-\xi3\phi}}\\
=\;& -\frac{\left(e^{\xi3\phi/2}-e^{-\xi3\phi/2}\right)\left(e^{\xi3\phi/2}+e^{-\xi3\phi/2}\right)\left(e^{\xi3\phi N}-e^{-\xi3\phi N}\right)}{2-e^{\xi3\phi}-e^{-\xi3\phi}}\\
=\;& \frac{\left(e^{\xi3\phi/2}-e^{-\xi3\phi/2}\right)\left(e^{\xi3\phi/2}+e^{-\xi3\phi/2}\right)\left(e^{\xi\phi N}-e^{-\xi\phi N}\right)\left(e^{2\xi\phi N}+e^{-2\xi\phi N}+1\right)}{\left(e^{\xi3\phi/2}-e^{-\xi3\phi/2}\right)^2}\\
=\;& \frac{\left(e^{\xi3\phi/2}+e^{-\xi3\phi/2}\right)}{\left(e^{\xi3\phi/2}-e^{-\xi3\phi/2}\right)}\left(e^{\xi\phi N}-e^{-\xi\phi N}\right)\left(e^{2\xi N\phi}-2+e^{-2\xi N\phi}+3\right)\\
=\;& \frac{\left(e^{\xi3\phi}+e^{-\xi3\phi}+2\right)\left(e^{\xi\phi N}-e^{-\xi\phi N}\right)\left(\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2U_{N-1}(k)^2+3\right)}{e^{\xi3\phi}-e^{-\xi3\phi}}\\
=\;& \frac{\left(\left(e^{\xi\phi}+e^{-\xi\phi}\right)^3-3\left(e^{\xi\phi}+e^{-\xi\phi}\right)+2\right)U_{N-1}(k)\left(\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2U_{N-1}(k)^2+3\right)}{\left(\left(e^{\xi\phi}+e^{-\xi\phi}\right)^2-1\right)}\\
=\;& \frac{\left((2k)^3-3(2k)+2\right)U_{N-1}(k)\left(\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2\left(U_{N-1}(k)\right)^2+3\right)}{(2k)^2-1}\\
=\;& 2\frac{(k+1)(2k-1)^2U_{N-1}(k)\left(\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2\left(U_{N-1}(k)\right)^2+3\right)}{(2k-1)(2k+1)}\\
=\;& 2\frac{(k+1)(2k-1)U_{N-1}(k)\left(\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2\left(U_{N-1}(k)\right)^2+3\right)}{2k+1}
\end{aligned}
$$

Putting the two parts together yields:

$$
\begin{aligned}
&\sum_{n=0}^{N-1} \left(T_{n+1}(k)\right)^3 + \left(T_n(k)\right)^3 \\
= \;& \frac{1}{8}\left(6(k+1)U_{N-1}(k) + 2\frac{(k+1)(2k-1)U_{N-1}(k)\left(\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2\left(U_{N-1}(k)\right)^2+3\right)}{2k+1}\right) \\
= \;& \frac{1}{4}(k+1)U_{N-1}(k)\left(3\frac{2k+1}{2k+1} + \frac{(2k-1)\left(\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2\left(U_{N-1}(k)\right)^2+3\right)}{2k+1}\right) \\
= \;& \frac{1}{4}\frac{k+1}{2k+1}\left(\frac{3(2k+1)}{\left(U_{N-1}(k)\right)^2} + (2k-1)\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2 + \frac{3(2k-1)}{\left(U_{N-1}(k)\right)^2}\right)\left(U_{N-1}(k)\right)^3 \\
= \;& \frac{1}{4}\frac{k+1}{2k+1}\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2\left(\frac{12k}{\left(U_{N-1}(k)\right)^2\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} + 2k-1\right)\left(U_{N-1}(k)\right)^3 \\
= \;& \frac{(k+1)^2(k-1)}{2k+1}\left(2k-1+\frac{12k}{\left(U_{N-1}(k)\right)^2\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2}\right)\left(U_{N-1}(k)\right)^3
\end{aligned}
$$

where the last equality follows because $\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2 = \left(e^{\xi\phi}+e^{-\xi\phi}\right)^2 - 4$. Then the preliminary result follows from equation (A.11). Let us now prove the statement of the proposition.

(1) If $\rho < 3/4$, no information can be communicated. In this case the controlling investor would choose action $1/2\theta_C$ (see Proposition A.1) giving rise to an expected payoff which, by Lemma B.1, equals

$$
-\frac{\sigma_X^2}{2r_E}\mathbb{E}\left(\frac{r_E\theta_E}{2\theta_C}-\omega\right)^2 = -\frac{\sigma_X^2}{2r_E}\mathbb{E}\left(\frac{1}{2\rho}-\omega\right)^2.
$$

(2) In the case $3/4 \leq \rho \leq 1$, we have $|k_\rho| \leq 1$ and substituting the relevant expression from expression (A.15) into (A.14) we get

$$
\begin{aligned}
V_E(\rho,N) &= -\frac{\sigma_X^2}{6r_E(k_\rho+1)}\frac{(k_\rho-1)}{2k_\rho+1}\left(2k_\rho-1-3\frac{k_\rho}{\sin^2(N\arccos k_\rho)}\right) \\
&= -\frac{\sigma_X^2}{6r_E}\frac{1-\rho}{\rho(4\rho-1)}\left(3\frac{2\rho-1}{\sin^2(N\arccos k_\rho)}+3-4\rho\right),
\end{aligned}
$$

where the last equality follows from substituting $2\rho-1$ for k_ρ and collecting terms. When $\sin^2 = 1$ the expression reduces to $\overline{V}_E(\rho)$. We have:

$$
\begin{aligned}
V_E(\rho,N) - V_E(\rho,N-1) &= -\frac{\sigma_X^2}{6r_E}\frac{1-\rho}{\rho(4\rho-1)}\left(3\frac{2\rho-1}{\sin^2(N\arccos k_\rho)} - 3\frac{2\rho-1}{\sin^2((N-1)\arccos k_\rho)}\right) \\
&= -\frac{\sigma_X^2}{2r_E}\frac{(1-\rho)(2\rho-1)}{\rho(4\rho-1)}\left(\frac{1}{\sin^2(N\arccos k_\rho)} - \frac{1}{\sin^2((N-1)\arccos k_\rho)}\right).
\end{aligned}
$$

The factor outside the parenthesis is negative as in Proposition A.2 part 1, and the term in parenthesis is the same, so monotonicity in N and tightness of the bound are proved as in that Proposition.

(3) In the case $\rho \geq 1$ we have $|k_\rho| \geq 1$, and substituting the relevant expression from expression (A.15) into (A.14) we get

$$\begin{aligned} V_E(\rho, N) &= -\frac{\sigma_X^2}{6r_E(k_\rho+1)}\frac{(k_\rho-1)}{2k_\rho+1}\left(2k_\rho - 1 + 3\frac{k_\rho}{\sinh^2(N \operatorname{arccosh} k_\rho)}\right) \\ &= -\frac{\sigma_X^2}{6r_E}\frac{(\rho-1)}{\rho(4\rho-1)}\left(3\frac{2\rho-1}{\sinh^2(N \operatorname{arccosh} k_\rho)} - 3 + 4\rho\right), \end{aligned}$$

where the last equality follows from substituting $2\rho - 1$ for k_ρ and collecting terms. The upper bound $\overline{V}_E(\rho)$ is approached for $N \to \infty$.

□

Lemma A.4 (Non-controlling investor's demand)**.** *Let $a^*(\Omega(\rho))$ denote the action chosen by the controlling agent if players are playing the most informative cheap talk equilibrium given ρ. Then:*

(1) *the demand function of a non-controlling investor with risk aversion r_i equals:*

$$\sigma_X^2(r_C\theta_C - r_i\theta_i)\mathbb{E}[a^*(\Omega)]^2;$$

(2) *for $\rho > 1$ we have:*

$$\mathbb{E}\left[(a^*(\Omega(\rho)))^2\right] = \frac{1}{\theta_C^2}\frac{\rho}{4\rho-1};$$

(3) *the non-controlling investor's demand function increases if the expert investor's preferences become more aligned with the controlling investor's preferences, i.e., if θ_E gets closer to θ_C/r_E.*

Proof. 1. Denote for simplicity $\Omega(\rho) = \Omega(\rho)$. The demand function of a non-controlling investor with risk aversion r_i equals:

$$\begin{aligned} -\frac{\partial}{\partial\theta_i}\frac{\sigma_X^2}{2r_i}\mathbb{E}[r_i\theta_i a^*(\omega) - \omega]^2 &= -\sigma_X^2\mathbb{E}[a^*(\omega)(r_i\theta_i a^*(\omega) - \omega)] = \sigma_X^2\mathbb{E}[\mathbb{E}[a^*(\omega)(\omega - r_i\theta_i a^*(\Omega))|\Omega]] \\ &= \sigma_X^2\mathbb{E}[a^*(\Omega)\mathbb{E}[\omega - r_C\theta_C a^*(\Omega) + r_C\theta_C a^*(\Omega) - r_i\theta_i a^*(\Omega)|\Omega]]. \quad (A.16) \end{aligned}$$

Now, $a^*(\Omega)$ must solve the following maximization problem:

$$\max_a -\frac{\sigma_X^2}{2r_C}\mathbb{E}\left[(r_C\theta_C a-\omega)^2|\Omega\right],$$

whose first order conditions w.r.t. a read $\mathbb{E}[\omega - r_C\theta_C a^*(\Omega)|\Omega]=0$. Use this expression to simplify (A.16), then isolate $a^*(\Omega)$ to get the desired expression.

2.

$$\begin{aligned}\mathbb{E}\left[(a^*(\Omega(\rho)))^2\right] &= \sum_{n=0}^{N-1}(\omega_{n+1}-\omega_n)\left(\frac{\omega_n+\omega_{n+1}}{2\theta_C}\right)^2\\ &= \frac{1}{4\theta_C^2}\frac{1}{(U_{N-1}(k_\rho))^3}\sum_{n=0}^{N-1}(U_n(k_\rho)-U_{n-1}(k_\rho))(U_n(k_\rho)+U_{n-1}(k_\rho))^2, \end{aligned} \tag{A.17}$$

where the last line follows from Proposition A.1 part 3. Now using equation (A.9) we have:

$$\begin{aligned}&\sum_{n=0}^{N-1}(U_n(k)-U_{n-1}(k))(U_n(k)+U_{n-1}(k))^2\\
=&\sum_{n=0}^{N-1}\left(\frac{e^{\xi(n+1)\phi}-e^{-\xi(n+1)\phi}}{e^{\xi\phi}-e^{-\xi\phi}}-\frac{e^{\xi n\phi}-e^{-\xi n\phi}}{e^{\xi\phi}-e^{-\xi\phi}}\right)\left(\frac{e^{\xi(n+1)\phi}-e^{-\xi(n+1)\phi}}{e^{\xi\phi}-e^{-\xi\phi}}+\frac{e^{\xi n\phi}-e^{-\xi n\phi}}{e^{\xi\phi}-e^{-\xi\phi}}\right)^2\\
=&\sum_{n=0}^{N-1}\left(\frac{1}{e^{\xi\phi}-e^{-\xi\phi}}\right)^3\left(e^{\xi(n+1)\phi}-e^{\xi n\phi}+e^{-\xi n\phi}-e^{-\xi(n+1)\phi}\right)\left(e^{\xi(n+1)\phi}+e^{\xi n\phi}-e^{-\xi n\phi}-e^{-\xi(n+1)\phi}\right)^2\\
=&\sum_{n=0}^{N-1}\frac{\left(e^{\xi\phi}-1\right)\left(e^{\xi\phi}+1\right)^2}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^3}\left(e^{\xi n\phi}+e^{-\xi(n+1)\phi}\right)\left(e^{\xi n\phi}-e^{-\xi(n+1)\phi}\right)^2\\
=&\frac{\left(e^{\xi\phi}-1\right)\left(e^{\xi\phi}+1\right)^2}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^3}\sum_{n=0}^{N-1}\left(e^{2\xi n\phi}-e^{-2\xi(n+1)\phi}\right)\left(e^{\xi n\phi}-e^{-\xi(n+1)\phi}\right)\\
=&\frac{\left(e^{\xi\phi}-1\right)\left(e^{\xi\phi}+1\right)^2}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^3}\sum_{n=0}^{N-1}\left(e^{3\xi n\phi}-e^{2\xi n\phi}e^{-\xi(n+1)\phi}-e^{\xi n\phi}e^{-2\xi(n+1)\phi}+e^{-3\xi(n+1)\phi}\right)\\
=&\frac{\left(e^{\xi\phi}-1\right)\left(e^{\xi\phi}+1\right)^2}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^3}\left[\frac{e^{3\xi N\phi}-1}{e^{3\xi\phi}-1}-\frac{e^{-2\xi\phi}-e^{\xi(N-2)\phi}}{e^{-\xi\phi}-1}-\frac{e^{-\xi(N+2)\phi}-e^{-2\xi\phi}}{e^{-\xi\phi}-1}+\frac{1-e^{-3\xi N\phi}}{e^{3\xi\phi}-1}\right]\\
=&\frac{\left(e^{\xi\phi}-1\right)\left(e^{\xi\phi}+1\right)^2}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^3}\left[\frac{e^{3\xi N\phi}-e^{-3\xi N\phi}}{e^{3\xi\phi}-1}-\frac{e^{-\xi(N+2)\phi}-e^{\xi(N-2)\phi}}{e^{-\xi\phi}-1}\right].\end{aligned}$$

We will work on computing the two parts separately. First:

$$\begin{aligned}&\frac{\left(e^{\xi\phi}-1\right)\left(e^{\xi\phi}+1\right)^2}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^3}\cdot\frac{e^{3\xi N\phi}-e^{-3\xi N\phi}}{e^{3\xi\phi}-1}\\
=&\frac{e^{2\xi\phi}+2e^{\xi\phi}+1}{e^{2\xi\phi}+e^{\xi\phi}+1}\cdot\frac{\left(e^{\xi N\phi}-e^{-\xi N\phi}\right)\left(e^{2\xi N\phi}+1+e^{-2\xi N\phi}\right)}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^3}\end{aligned}$$

$$
\begin{aligned}
&= \frac{e^{\xi\phi}+2+e^{-\xi\phi}}{e^{\xi\phi}+1+e^{-\xi\phi}} U_{N-1}(k) \frac{\left(e^{2\xi N\phi}-2+e^{-2\xi N\phi}\right)+3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} \\
&= \frac{2+2k}{1+2k} U_{N-1}(k) \left(\left(U_{N-1}(k)\right)^2 + \frac{3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} \right).
\end{aligned}
$$

The second part is:

$$
\begin{aligned}
\frac{\left(e^{\xi\phi}-1\right)\left(e^{\xi\phi}+1\right)^2}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^3} \cdot \frac{e^{-\xi(N+2)\phi}-e^{\xi(N-2)\phi}}{e^{-\xi\phi}-1} &= -\frac{\left(e^{\xi\phi}-1\right)\left(e^{\xi\phi}+1\right)^2 e^{-2\xi\phi}}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^3} \frac{e^{\xi N\phi}-e^{-\xi N\phi}}{e^{-\xi\phi}-1} \\
&= \frac{\left(1-e^{\xi\phi}\right)\left(e^{\xi\phi}+1\right)^2 e^{-2\xi\phi}}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2\left(e^{-\xi\phi}-1\right)} U_{N-1}(k) \\
&= \frac{\left(e^{\xi\phi}+1\right)^2 e^{-\xi\phi}}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} U_{N-1}(k) \\
&= \frac{2k+2}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} U_{N-1}(k)
\end{aligned}
$$

where the last step follows from equation (A.10). Putting the two parts together, we have that:

$$
\begin{aligned}
&\sum_{n=0}^{N-1} \left(U_n(k)-U_{n-1}(k)\right)\left(U_n(k)+U_{n-1}(k)\right)^2 \\
&= \frac{2+2k}{1+2k} U_{N-1}(k) \left(U_{N-1}(k)^2 + \frac{3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} \right) - \frac{2k+2}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} U_{N-1}(k) \\
&= \frac{2+2k}{1+2k} U_{N-1}(k) \left(U_{N-1}(k)^2 + \frac{3}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} - \frac{1+2k}{\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} \right) \\
&= \frac{2+2k}{1+2k} U_{N-1}(k)^3 \left(1 - 2\frac{1+k}{\left(U_{N-1}(k)\right)^2\left(e^{\xi\phi}-e^{-\xi\phi}\right)^2} \right) \\
&= \frac{2+2k}{1+2k} U_{N-1}(k)^3 \left(1 - 2\frac{1+k}{\left(e^{\xi N\phi}-e^{-\xi N\phi}\right)^2} \right) \\
&= \frac{2+2k}{1+2k} \left(1 - \frac{1+k}{2\sinh^2\left(N \operatorname{arccosh} k\right)} \right) \left(U_{N-1}(k)\right)^3,
\end{aligned}
$$

where the last equality uses equation (A.11). Plugging this expression back into (A.17), and noting that when $\rho > 1$ the optimal number of partitions is $N \to \infty$, we get:

$$
\mathbb{E}\left[\left(a^*\left(\Omega\left(\rho\right)\right)\right)^2\right] = \frac{1}{\theta_C^2}\frac{\rho}{4\rho-1} = \frac{1}{\theta_C\left(4\theta_C - r_E\theta_E\right)},
$$

after substituting $2\rho-1$ for k_ρ and collecting terms. This expression is increasing in θ_E if $r_E\theta_E \le \theta_C$, hence the non-controlling investor's demand function increases if the expert investor's preferences become more aligned with the controlling investor's preferences. □